E. Breuillard Michaelmas 2019

Probability and Measure 4

*Exercises marked with a star * are not examinable*

1. (a) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$. Show that $\mathbb{L}^2(\Omega, \mathcal{G}, \mathbb{P})$ is a closed subspace of $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P})$.

(b) Let $V_1 \leq V_2 \leq \ldots$ be an increasing sequence of closed subspaces of $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. For $f \in \mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, denote by f_n the orthogonal projection of f on V_n. Show that f_n converges in $\mathbb{L}^2$.

2. (continuity of translation in $\mathbb{L}^p$). Let $p \in [1, +\infty)$ and $f \in \mathbb{L}^p(\mathbb{R}^d)$. For $t \in \mathbb{R}^d$ let $\tau_t(f)$ be the translate $\tau_t(f)(x) = f(x+t)$. Show that $\tau_t(f) \in \mathbb{L}^p(\mathbb{R}^d)$ and that the map $t \mapsto \tau_t(f)$ is continuous from $\mathbb{R}^d$ to $\mathbb{L}^p(\mathbb{R}^d)$. What happens when $p = +\infty$?

3. Let μ be a Borel probability measure on $\mathbb{R}^d$ and let $(\mu_n : n \in \mathbb{N})$ be a sequence of such measures. Suppose that $\mu_n(f) \to \mu(f)$ for all C^∞ functions on $\mathbb{R}^d$ of compact support. Show that μ_n converges weakly to μ on $\mathbb{R}^d$.

4. For a finite Borel measure μ on the line show that, if $\int |x|^k d\mu(x) < \infty$, then the Fourier transform $\hat{\mu}$ of μ has a kth continuous derivative, which at 0 is given by

$$\hat{\mu}^{(k)}(0) = i^k \int x^k d\mu(x).$$

Show further that if X, Y are two bounded random variables such that $\mathbb{E}(X^k) = \mathbb{E}(Y^k)$ for all integers $k \geq 0$, then X and Y have the same distribution.

5. (i) Show that for any real numbers a, b one has $\int_a^b e^{itx} dx \to 0$ as $|t| \to \infty$.

(ii) Show that, for any $f \in L^1(\mathbb{R})$, the Fourier transform

$$\hat{f}(t) = \int_{-\infty}^{\infty} e^{itx} f(x) dx$$

tends to 0 as $|t| \to \infty$. This is the *Riemann–Lebesgue Lemma.*

6. Let g be a C^∞ function of compact support on $\mathbb{R}^d$. Show that $\hat{g}$ is integrable.

7. The Schwartz space $\mathcal{S}$ of $\mathbb{R}$ is the space of all C^∞ complex valued functions f on $\mathbb{R}$ such that for every $k, n \in \mathbb{Z}_{\geq 0}$ we have $f^{(k)}(x) = O(1/|x|^n)$ as $|x| \to +\infty$. Show that if f belongs to $\mathcal{S}$ so does $\hat{f}$.

8. Let $X_1, \dots, X_n$ be independent $N(0,1)$ gaussian random variables. Show that

$$\left(\overline{X}, \sum_{m=1}^{n} (X_m - \overline{X})^2\right) \quad \text{and} \quad \left(\frac{X_n}{\sqrt{n}}, \sum_{m=1}^{n-1} X_m^2\right)$$

have the same distribution, where $\overline{X} = (X_1 + \cdots + X_n)/n$.

9. The Cauchy distribution has density function $f(x) = \pi^{-1}(1 + x^2)^{-1}$ for $x \in \mathbb{R}$. Show that one can simulate a random variable X whose law follows the Cauchy distribution by picking a random angle θ uniformly in $(-\frac{\pi}{2}, \frac{\pi}{2})$ and setting $X = \tan\theta$. Show that the corresponding characteristic function is given by $\varphi(u) = e^{-|u|}$. Show also that, if $X_1, \dots, X_n$ are independent Cauchy random variables, then the random variable $(X_1 + \cdots + X_n)/n$ is also Cauchy.

11. Let $(X, \mathcal{A}, \mu, T)$ be a measure-preserving system.

(a) Show that it is ergodic if and only if every T-invariant measurable function is constant almost everywhere.

(b) If f is a measurable function on X such that $f = f \circ T$ almost everywhere, show that there is a T-invariant measurable function g on X such that $g = f$ almost everywhere.

12. Let $(X_n)_n$ be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Assume that $\mathcal{F} = \sigma(X_1, X_2, \dots)$. Show that for every $\epsilon > 0$, and every $A \in \mathcal{F}$, there is $n \geq 1$ and $B \in \mathcal{F}_n$ such that $\mathbb{P}(A \triangle B) < \epsilon$.

13. (Markov shift) Let $p \in (0,1)$ and $q = 1-p$. Consider a random sequence of 0's and 1's, where the first digit is chosen uniformly at random (i.e. $\mathbb{P}(X_1 = 0) = \mathbb{P}(X_1 = 1) = \frac{1}{2}$) and at each step the next digit either remains the same with probability p, or changes with probability q. Let $(X_n)_{n\geq 1}$ be the random process thus obtained.

(1) Show that $(X_n)_{n\geq 1}$ is a stationary process, and let $(X, \mathcal{A}, \mu, T)$ be the associated canonical model, where $X = \{0,1\}^{\mathbb{N}}$, $\mathcal{A}$ the σ-algebra generated by cylinders and T is the shift.

(2) Show that for any two cylinder sets A, B we have

$$\mu(A \cap T^{-k}B) \to_{k\to+\infty} \mu(A)\mu(B).$$

(3) Prove that the previous limit holds for all $A, B \in \mathcal{A}$ and deduce that $(X, \mathcal{A}, \mu, T)$ is ergodic.

14. Let $(X, \mathcal{A}, \mu, T)$ be an ergodic measure preserving system, $\mathcal{H} = \mathbb{L}^2(X, \mathcal{A}, \mu)$ and $U : \mathcal{H} \to \mathcal{H}$ the operator $f \mapsto f \circ T$. Show that every eigenvalue of U is simple (i.e. its space of eigenfunctions is one dimensional) and that the set of eigenvalues of U forms a subgroup of $\{z \in \mathbb{C}, |z| = 1\}$.

15.*. (Sobolev embedding) Say that $f \in \mathbb{L}^2(\mathbb{R})$ is $\mathbb{L}^2$-differentiable if there is a function Df in $\mathbb{L}^2(\mathbb{R})$ (called the $\mathbb{L}^2$-derivative of f) such that

$$\|\theta_h f - f - hDf\|_2/h \to 0 \quad \text{as} \quad h \to 0,$$

where $\theta_h f(x) = f(x+h)$. For example show that the function $f(x) = \max(1-|x|, 0)$ is $\mathbb{L}^2$-differentiable and find its $\mathbb{L}^2$-derivative.

Suppose that $f \in \mathbb{L}^1(\mathbb{R}) \cap \mathbb{L}^2(\mathbb{R})$ is $\mathbb{L}^2$-differentiable. Show that f has admits a continuous version (i.e. $\exists g$ continuous s.t. $g(x) = f(x)$ a.e.).

Hint: show that $u\hat{f}(u) \in \mathbb{L}^2$ and that $\|f\|_\infty \leq C\|(1+|u|)\hat{f}(u)\|_2$ for some finite absolute constant C, to be determined. This is the Sobolev embedding theorem and Sobolev inequality.

16.*. Let $X = (X_1, \ldots, X_n)$ be a Gaussian random variable in $\mathbb{R}^n$ with mean μ and covariance matrix V. Assume that V is invertible write $V^{-1/2}$ for the positive-definite square root of V^{-1}. Set $Y = (Y_1, \ldots, Y_n) = V^{-1/2}(X - \mu)$. Show that $Y_1, \ldots, Y_n$ are independent $N(0,1)$ random variables. Show further that we can write X_2 in the form $X_2 = aX_1 + Z$ where Z is independent of X_1 and determine the distribution of Z.

17.*. For each $n \in \mathbb{N}$, there is a unique probability measure μ_n on the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ such that $\mu_n(A) = \mu_n(UA)$ for all Borel sets A and all orthogonal $n \times n$ matrices U. Fix $k \in \mathbb{N}$ and, for $n \geq k$, let γ_n denote the probability measure on $\mathbb{R}^k$ which is the law of $\sqrt{n}(x^1, \ldots, x^k)$ under μ_n. Show

(a) if $X \sim N(0, I_n)$ then $X/|X| \sim \mu_n$,

(b) if $(X_n : n \in \mathbb{N})$ is a sequence of independent $N(0,1)$ random variables and if $R_n = \sqrt{X_1^2 + \cdots + X_n^2}$ then $R_n/\sqrt{n} \to 1$ a.s.,

(c) γ_n converges weakly to the standard Gaussian distribution on $\mathbb{R}^k$ as $n \to \infty$.