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Probability and Measure 3

*Exercises marked with a star * are not examinable*

1. A coin is tossed infinitely often, making an infinite sequence $\omega_1, \ldots, \omega_n, \ldots$ of heads or tails, i.e. $\omega_i \in \{H, T\}$. Show that every finite sequence of heads and tails (such as $HHTTTHT$) occurs infinitely often almost surely.

2. Let $(A_n : n \in \mathbb{N})$ be a sequence of events, with $\mathbb{P}(A_n) = 1/n^2$ for all n. Set $X_n = n^2 1_{A_n} - 1$ and set $\bar{X}_n = (X_1 + \cdots + X_n)/n$. Show that $\mathbb{E}(\bar{X}_n) = 0$ for all n, but that $\bar{X}_n \to -1$ almost surely as $n \to \infty$.

3. (Weak law of large numbers) Let $\{X_n\}_{n\geq 1}$ be a sequence of real random variables, such that $\mathbb{E}(|X_n|^2) < \infty$ for each n and $\sum_{k=1}^n \mathbb{E}(X_k^2) = o(n^2)$ as $n \to +\infty$. Assume further that $\mathbb{E}(X_n) = 0$ for all n and that the variables are pairwise uncorrelated, i.e. $\mathbb{E}(X_i X_j) = 0$ if $i \neq j$. Show that $\frac{1}{n}\sum_{k=1}^n X_k$ converges to 0 in probability.

4. Let μ, $\{\mu_n\}_{n\geq 1}$ be Borel probability measures on $\mathbb{R}$ with distribution functions F and $\{F_n\}_{n\geq 1}$ respectively. Show that μ_n converges weakly to μ if and only if $F_n(x)$ converges to $F(x)$ for every real x, where F is continuous, and also if and only if $F_n(x)$ converges to $F(x)$ for Lebesgue almost every $x \in \mathbb{R}$.

5. (de Moivre-Laplace) Let X_n be a binomial random variable $B(n, \frac{1}{2})$, e.g. X_n is the number of heads obtained after tossing a fair coin n times. Use the Stirling formula ($n! e^n n^{-n-\frac{1}{2}} \to \sqrt{2\pi}$) to show that

$$\sqrt{n}\mathbb{P}(X_n = k) = 2e^{-2(k-n/2)^2/n}/\sqrt{2\pi} + o(1)$$

as $n \to +\infty$ uniformly over k when $(k - n/2)/\sqrt{n}$ remains bounded. Deduce that $(X_n - \mathbb{E}(X_n))/\sqrt{n}$ converges in distribution to a gaussian $\mathcal{N}(0, \sigma^2)$ with $\sigma^2 = \frac{1}{4}$.

6. Let $(X_n : n \in \mathbb{N})$ be a sequence of independent random variables, such that $\mathbb{E}(X_n) = \mu$ and $\mathbb{E}(X_n^4) \leq M$ for all n, for some constants $\mu \in \mathbb{R}$ and $M < \infty$. Set $P_n = X_1X_2+X_2X_3+\cdots+X_{n-1}X_n$. Show that P_n/n converges a.s. as $n \to \infty$ and identify the limit.

7. Let $\mu, \{\mu_n\}_{n\geq 1}$ be Borel probability measures on $\mathbb{R}$ and assume that μ_n converges weakly to μ. Show that one can find some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random variables $X, \{X_n\}_{n\geq 1}$ such that X has law μ, X_n has law μ_n and $X_n \to X$ almost surely as $n \to +\infty$. Can the X_n be chosen to be independent ?

8. Let $(X_n : n \in \mathbb{N})$ be an identically distributed sequence with $\mathbb{E}(|X_1|^2) < \infty$. Show that $n\mathbb{P}(|X_1| > \varepsilon\sqrt{n}) \to 0$ as $n \to \infty$, for all $\varepsilon > 0$. Deduce that $n^{-1/2}\max_{k\leq n}|X_k| \to 0$ in probability. And that

$$\mathbb{E}(\max_{k\leq n}|X_k|)/\sqrt{n} \to 0 \text{ as } \quad n \to \infty.$$

9. Find a uniformly integrable sequence of random variables $(X_n : n \in \mathbb{N})$ such that both $X_n \to 0$ a.s. and $\mathbb{E}\big(\sup_n |X_n|\big) = \infty$.

10. (2nd+ Borel-Cantelli) Let $\{A_n\}_{n\geq 1}$ be a sequence of events. Assume that $\sum_{n\geq 1}\mathbb{P}(A_n) = +\infty$.

(i) first, assume that the events are pairwise independent in the sense that $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j)$ for every two distinct $i \neq j$. Show that $\mathbb{P}(\limsup A_n) = 1$.

(ii) next, assume only that the events are pairwise weakly independent in the sense that there is some $C \geq 1$ such that $\mathbb{P}(A_i \cap A_j) \leq C\mathbb{P}(A_i)\mathbb{P}(A_j)$ for every two distinct $i \neq j$. Show that $\mathbb{P}(\limsup A_n) > 0$.

Hint: Let $S_n = \sum_{k=1}^n 1_{A_k}$, *for (i) use Chebychev's inequality as in Exercise 3 to estimate* $\mathbb{P}(\sup_n S_n \leq k) \leq \mathbb{P}(|S_n - \mathbb{E}(S_n)| \geq \mathbb{E}(S_n) - k)$, *for (ii) and show that* $\frac{S_n}{\mathbb{E}(S_n)}$ *is bounded in* $\mathbb{L}^2$ *hence uniformly integrable.*

11. Let $X_1, \dots, X_n$ be n real random variables with $\mathbb{E}(|X_i|^2) < \infty$ for $i = 1, \dots, n$. The covariance matrix $\mathrm{var}(X) = (c_{ij} : 1 \leq i, j \leq n)$ of X is defined by

$$c_{ij} = \mathrm{cov}(X_i, X_j) := \mathbb{E}[(X_i - \mathbb{E}(X_i))(X_j - \mathbb{E}(X_j)].$$

Show that $\mathrm{var}(X)$ is a non-negative definite matrix.

12. Let X be a random variable and let $1 \leq p < \infty$. Show that, if $X \in L^p(\mathbb{P})$, then $\mathbb{P}(|X| \geq \lambda) = O(\lambda^{-p})$ as $l \to \infty$. Prove the identity

$$\mathbb{E}(|X|^p) = \int_0^\infty p\lambda^{p-1}\mathbb{P}(|X| \geq \lambda)d\lambda$$

and deduce that, for all $q > p$, if $\mathbb{P}(|X| \geq \lambda) = O(\lambda^{-q})$ as $l \to \infty$, then $X \in L^p(\mathbb{P})$.

13. Define a function f on $\mathbb{R}$ by setting $f(x) = \exp(-1/x)$ for $x > 0$ and $f(x) = 0$ otherwise. Show that f is C^∞. Now let $\phi(x) = f(x)f(1-x)$ and $\psi(x) = c^{-1}\int_{-\infty}^x \phi(t)dt$, where $c = \int_{\mathbb{R}} \phi(t)dt$. Check that ψ is C^∞, $\psi(x) = 0$ if $x \leq 0$, $\psi(x) = 1$ if $x \geq 1$ and ψ is non-decreasing. Use ψ to build, for each interval $I = [a, b]$ and $\epsilon > 0$ a C^∞-function $\psi_{I,\epsilon}$ on $\mathbb{R}$ such that

$$1_{[a,b]} \leq \psi_{I,\epsilon} \leq 1_{[a-\epsilon,b+\epsilon]}.$$

Use this to construct for each compact set $K \subset \mathbb{R}^d$ and each open set $U \supset K$ a C^∞-function $\psi_{K,U}$ on $\mathbb{R}^d$ such that

$$1_K \leq \psi_{K,U} \leq 1_U.$$

Deduce that the smooth functions of compact support $C_c^\infty(\mathbb{R}^d)$ form a dense subspace of $L^p(\mathbb{R}^d)$ for any $p \in [1, +\infty)$.

14.*. Let μ be a Borel probability measure on $\mathbb{R}^d$. Show that there is a sequence of finitely supported probability measures μ_n on $\mathbb{R}^d$, which converges weakly to μ.

15.*. Let μ and $\{\mu_n\}_{n\geq 1}$ be Borel probability measures on $\mathbb{R}^d$. Show that the following are equivalent:

(1) μ_n converges weakly to μ as $n \to \infty$.
(2) for every Borel set A such that $\mu(\partial A) = 0$ we have $\mu_n(A) \to \mu(A)$ as $n \to \infty$ (here ∂A is the boundary of A, i.e. the points in the closure of A that are not in the interior of A.)