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Probability and Measure 1

*Exercises marked with a star * are not examinable*

1. Let E be a Lebesgue measurable subset of the real line with positive Lebesgue measure $m(E)$. Show that for every $\epsilon > 0$ there exists an open interval (a, b) such that $m(E \cap (a, b)) > (1 - \epsilon)|a - b|$.

2. Show that the following sets of subsets of $\mathbb{R}$ all generate the same σ-algebra:
(a) $\{(a, b) : a < b\}$, (b) $\{(a, b] : a < b\}$, (c) $\{(-\infty, b] : b \in \mathbb{R}\}$.

3. Let E be a set and let $\mathcal{S}$ be a set of σ-algebras on E. Define

$$\mathcal{E}^* = \{A \subseteq E : A \in \mathcal{E} \quad \text{for all} \quad \mathcal{E} \in \mathcal{S}\}.$$

Show that $\mathcal{E}^*$ is a σ-algebra on E. Show, on the other hand, by example, that the union of two σ-algebras on the same set need not be a σ-algebra.

4. Let E be a set and $\mathcal{B}$ a Boolean algebra of subsets of E. Let $m : \mathcal{B} \to [0, +\infty]$ be such that $m(\varnothing) = 0$. Suppose that m is finitely additive. Show that m is countably additive if and only if it is countably subadditive.

5. Let E be a set and $\mathcal{E}$ a family of subsets of E, which contains E and $\varnothing$, and is stable under complementation, under countable disjoint unions and under finite intersections. Show that $\mathcal{E}$ is a σ-algebra.

6. Let X be a set and $\mathcal{A}$ a Boolean algebra of subsets of X. Let $\mu : \mathcal{A} \to [0, +\infty)$ be a finitely additive measure. Show that μ is countably additive on $\mathcal{A}$ (i.e. $\mu(\bigcup A_n) = \sum_n \mu(A_n)$ provided the $A_n \in \mathcal{A}$ are disjoint and $\bigcup A_n \in \mathcal{A}$) if and only if the following "continuity property" holds: for any decreasing sequence $(A_n : n \in \mathbb{N})$ of sets in $\mathcal{A}$, with $\cap_n A_n = \emptyset$, we have $\mu(A_n) \to 0$.

7. Let $(E, \mathcal{E}, \mu)$ be a finite measure space. Recall that for any sequence of sets $(A_n : n \in \mathbb{N})$ in $\mathcal{E}$, $\liminf A_n$ is the subset of those $x \in E$ such that $x \in A_m$ for all large enough $m \in \mathbb{N}$, and $\limsup A_n$ is the subset of those $x \in E$ such that x belongs to A_m for infinitely many $m \in \mathbb{N}$. Show that

$$\mu(\liminf A_n) \leq \liminf \mu(A_n) \leq \limsup \mu(A_n) \leq \mu(\limsup A_n).$$

Show that the first inequality remains true without the assumption that $\mu(E) < \infty$, but that the last inequality may then be false.

8. Let $(X, \mathcal{A})$ be a measurable space. Suppose that a function f on X has two representations

$$f = \sum_{k=1}^{m} a_k 1_{A_k} = \sum_{j=1}^{n} b_j 1_{B_j},$$

where each A_k and B_j belong to $\mathcal{A}$ and $a_k, b_j \in [0, +\infty)$. Show that, for any measure μ,

$$\sum_{k=1}^{m} a_k \mu(A_k) = \sum_{j=1}^{n} b_j \mu(B_j).$$

hint: for $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in \{0,1\}^m$, define $A_\varepsilon = A_1^{\varepsilon_1} \cap \dots \cap A_m^{\varepsilon_m}$ where $A_k^0 = A_k^c$ and $A_k^1 = A_k$. Define similarly B_δ for $\delta \in \{0,1\}^n$. Then set $f_{\varepsilon,\delta} = \sum_{k=1}^m \varepsilon_k a_k$ if $A_\varepsilon \cap B_\delta \neq \emptyset$ and $f_{\varepsilon,\delta} = 0$ otherwise. Show then that

$$\sum_{k=1}^{m} a_k \mu(A_k) = \sum_{\varepsilon,\delta} f_{\varepsilon,\delta} \mu(A_\varepsilon \cap B_\delta)$$

9. Recall that the outer measure $m^*(E)$ of a subset E of $\mathbb{R}^d$ is defined as

$$m^*(E) = \inf \sum_n m(B_n)$$

where the infimum is taken over all covers of E by countable unions $\bigcup_{n\in\mathbb{N}} B_n$ of boxes $B_n \subset \mathbb{R}^d$, and $m(B_n)$ is the product of the side lengths of the box B_n.

Let E be a subset of $X := [0,1]^d$. In Lebesgue's 1901 original article, E is defined to be (Lebesgue) measurable if $m^*(E) + m^*(X \setminus E) = 1$. Show that this definition equivalent to the one given in class.

10. Let $(E, \mathcal{E}, \mu)$ be a measure space. Call a subset $N \subseteq E$ *null* if $N \subseteq B$ for some $B \in \mathcal{E}$ with $\mu(B) = 0$. Write $\mathcal{N}$ for the set of null sets. Prove that the set of subsets $\mathcal{E}^\mu = \{A \cup N : A \in \mathcal{E}, N \in \mathcal{N}\}$ is a σ-algebra and show that μ has a well-defined and countably additive extension to $\mathcal{E}^\mu$ given by $\mu(A \cup N) = \mu(A)$. We call $\mathcal{E}^\mu$ the *completion of* $\mathcal{E}$ *with respect to* μ. Suppose now that E is σ-finite and write μ^* for the outer measure associated to μ, as in the proof of Carathéodory's Extension Theorem. Show that $\mathcal{E}^\mu$ is exactly the set of μ^*-measurable sets.

11. Let $X = \mathbb{R}^d$ endowed with the σ-algebra $\mathcal{B}$ of Borel sets. A Dirac mass at $x \in X$ is the measure δ_x on $\mathcal{B}$ such that $\delta_x(A) = 1$ or 0 according as $x \in A$ or $x \notin A$. Let μ be a positive linear combination of a finite number of Dirac masses. What is the completion of $\mathcal{B}$ with respect to μ ?

12. Let C_n denote the nth approximation to the Cantor set C: thus $C_0 = [0,1]$, $C_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$, $C_2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]$, etc. and $C_n \downarrow C$ as $n \to \infty$. Show that C is Lebesgue measurable and has measure 0. Note that $[0,1] \backslash C_n$ is a union of $2^n - 1$ open intervals $I_1, \ldots, I_{2^n-1}$ read from left to right. Let $F_n : [0,1] \to [0,1]$ be the function equal to the constant $k/2^n$ on the k-th open interval, which is defined to be linear in between, continuous on $[0,1]$ and with $F_n(0) = 0, F_n(1) = 1$. Show that $F_n(x)$ converges uniformly on $[0,1]$ to a function $F(x)$, which is differentiable with derivative 0 at Lebesgue almost every point in $[0,1]$.

Hint: express F_{n+1} recursively in terms of F_n and use this relation to obtain a uniform estimate on $F_{n+1} - F_n$.

13*. A subset $E \subset \mathbb{R}$ is called Jordan measurable if for every $\epsilon > 0$ there are two finite unions of intervals $A = \bigcup_1^n I_i$ and $B = \bigcup_1^m J_j$ such that $A \subset E \subset B$ and $m(B \setminus A) < \varepsilon$, where m is defined on finite disjoint unions of intervals as the total length of the intervals.

Give an example of a compact subset of $[0,1]$ that is not Jordan measurable.

14*. Recall that a subset $E \subset \mathbb{R}^d$ is called Jordan measurable if for every $\epsilon > 0$ there are two elementary sets $A = \bigcup_1^n B_i$ and $B = \bigcup_1^m B'_j$, where the B_i, B'_j are bounded boxes in $\mathbb{R}^d$, such that $A \subset E \subset B$ and $m(B \setminus A) < \varepsilon$, where m is the elementary measure defined on elementary sets.

Show that a bounded subset of $\mathbb{R}^d$ is Jordan measurable if and only if it is Lebesgue measurable and its boundary has Lebesgue measure zero.

15*. Let $a < b$ be real numbers and $f : [a, b] \to \mathbb{R}$ a function. We denote by $\mathcal{P}$ a (marked) subdivision $a = t_0 < t_1 < \ldots < t_n = b$ of the interval $[a, b]$ together with the choice of a point $x_i \in [t_{i-1}, t_i]$ for $i = 1, \ldots, n$. The quantity $\tau(\mathcal{P}) := \max_{1 \le i \le n} |t_i - t_{i-1}|$ is called the width of the subdivision. The *Riemann sum* $S_{\mathcal{P}}(f)$ is defined by:

$$S_{\mathcal{P}}(f) = \sum_{1 \le i \le n} f(x_i)(t_i - t_{i-1}).$$

One says that f is *Riemann integrable* if all Riemann sums, for varying $\mathcal{P}$, converge to the same limit as $\tau(\mathcal{P}) \to 0$. This limit is called the *Riemann integral* of f and is denoted by $\int_{[a,b]} f$.

Show that a subset $E \subset [a, b]$ is Jordan measurable if and only if the indicator function 1_E is Riemann integrable. Moreover in this case $m(E) = \int_{[a,b]} 1_E$.

16*. Let X be a set and $\mathcal{F}$ be a family of subsets of X. Consider all Boolean algebras of subsets of X containing $\mathcal{F}$ and let $\beta(\mathcal{F})$ be their intersection. Show that $\beta(\mathcal{F})$ is the Boolean algebra, whose subsets are finite unions of sets that are finite intersections of subsets F of X such that either F or its complement $X \setminus F$ lies in $\mathcal{F}$.

17*. Let X be a set. A *monotone class* $\mathcal{M}$ on X is a family of subsets of X which is stable under increasing countable unions and decreasing countable intersections. Show that an intersection of monotone classes is again a monotone class. Let now $\mathcal{B}$ be a Boolean algebra of subsets of X. Show the *monotone class theorem*: the smallest monotone class $\mathcal{M}$ containing $\mathcal{B}$ is the σ-algebra generated by $\mathcal{B}$.