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**Probability and Measure 3**

*Exercises marked with a star * are not examinable*

**1.** A coin is tossed infinitely often, making an infinite sequence $\omega_1, \dots, \omega_n, \dots$ of heads or tails, i.e. $\omega_i \in \{H, T\}$. Show that every finite sequence of heads and tails (such as $HHTTTHT$) occurs infinitely often almost surely.

**2.** (Weak law of large numbers) Let $\{X_n\}_{n\geq 1}$ be a sequence of real random variables, such that $\mathbb{E}(|X_n|^2) < \infty$ for each $n$ and $\sum_{k=1}^n \mathbb{E}(X_k^2) = o(n^2)$ as $n \to +\infty$. Assume further that $\mathbb{E}(X_n) = 0$ for all $n$ and that the variables are pairwise uncorrelated, i.e. $\mathbb{E}(X_i X_j) = 0$ if $i \neq j$. Show that $\frac{1}{n}\sum_{k=1}^n X_k$ converges to 0 in probability.

**3.** Let $\mu$, $\{\mu_n\}_{n\geq 1}$ be Borel probability measures on $\mathbb{R}$ with distribution functions $F$ and $\{F_n\}_{n\geq 1}$ respectively. Show that $\mu_n$ converges weakly to $\mu$ if and only if $F_n(x)$ converges to $F(x)$ for every real $x$, where $F$ is continuous, and also if and only if $F_n(x)$ converges to $F(x)$ for Lebesgue almost every $x \in \mathbb{R}$.

**4.** (de Moivre-Laplace) Let $X_n$ be a binomial random variable $B(n, \frac{1}{2})$, e.g. $X_n$ is the number of heads obtained after tossing a fair coin $n$ times. Use the Stirling formula ($n! e^n n^{-n-\frac{1}{2}} \to \sqrt{2\pi}$) to show that
$$\sqrt{n}\mathbb{P}(X_n = k) = 2e^{-2(k-n/2)^2/n}/\sqrt{2\pi} + o(1)$$
as $n \to +\infty$ uniformly over $k$ when $(k - n/2)/\sqrt{n}$ remains bounded. Deduce that $(X_n - \mathbb{E}(X_n))/\sqrt{n}$ converges in distribution to a gaussian $\mathcal{N}(0, \sigma^2)$ with $\sigma^2 = \frac{1}{4}$.

**5.** Let $(X_n : n \in \mathbb{N})$ be a sequence of independent random variables, such that $\mathbb{E}(X_n) = \mu$ and $\mathbb{E}(X_n^4) \leq M$ for all $n$, for some constants $\mu \in \mathbb{R}$ and $M < \infty$. Set $P_n = X_1X_2 + X_2X_3 + \cdots + X_{n-1}X_n$. Show that $P_n/n$ converges a.s. as $n \to \infty$ and identify the limit.

**6.** Let $\mu, \{\mu_n\}_{n\geq 1}$ be Borel probability measures on $\mathbb{R}$ and assume that $\mu_n$ converges weakly to $\mu$. Show that one can find some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random variables $X, \{X_n\}_{n\geq 1}$ such that $X$ has law $\mu$, $X_n$ has law $\mu_n$ and $X_n \to X$ almost surely as $n \to +\infty$. Can the $X_n$ be chosen to be independent ?

**7.** Let $(X_n : n \in \mathbb{N})$ be an identically distributed sequence with $\mathbb{E}(|X_1|^2) < \infty$. Show that $n\mathbb{P}(|X_1| > \varepsilon\sqrt{n}) \to 0$ as $n \to \infty$, for all $\varepsilon > 0$. Deduce that $n^{-1/2} \max_{k\leq n} |X_k| \to 0$ in probability. And that

$$\mathbb{E}(\max_{k\leq n} |X_k|)/\sqrt{n} \to 0 \text{ as } \quad n \to \infty.$$

**8.** Find a uniformly integrable sequence of random variables $(X_n : n \in \mathbb{N})$ such that both $X_n \to 0$ a.s. and $\mathbb{E}\big(\sup_n |X_n|\big) = \infty$.

**9.** Let $\{A_n\}_{n\geq 1}$ be a sequence of events, which are pairwise weakly independent in the sense that there is some $C \geq 1$ such that $\mathbb{P}(A_i \cap A_j) \leq C\mathbb{P}(A_i)\mathbb{P}(A_j)$ for every two distinct $i \neq j$. Assume that $\sum_{n\geq 1} \mathbb{P}(A_n) = +\infty$. Show that $\mathbb{P}(\limsup A_n) > 0$.

*Hint: Let* $S_n = \sum_{k=1}^n 1_{A_k}$ *and show that* $Y_n = \frac{S_n}{\mathbb{E}(S_n)}$ *is bounded in* $\mathbb{L}^2$ *hence uniformly integrable.*

**10.** Let $X$ be a random variable and let $1 \leq p < \infty$. Show that, if $X \in L^p(\mathbb{P})$, then $\mathbb{P}(|X| \geq \lambda) = O(\lambda^{-p})$ as $l \to \infty$. Prove the identity

$$\mathbb{E}(|X|^p) = \int_0^\infty p\lambda^{p-1}\mathbb{P}(|X| \geq \lambda)d\lambda$$

and deduce that, for all $q > p$, if $\mathbb{P}(|X| \geq \lambda) = O(\lambda^{-q})$ as $l \to \infty$, then $X \in L^p(\mathbb{P})$.

**11.** A *stepfunction* $f : \mathbb{R} \to \mathbb{R}$ is any finite linear combination of indicator functions of finite intervals. Show that the set of stepfunctions $\mathcal{I}$ is dense in $L^p(\mathbb{R})$ for all $p \in [1, \infty)$: that is, for all $f \in L^p(\mathbb{R})$ and all $\varepsilon > 0$ there exists $g \in \mathcal{I}$ such that $\|f - g\|_p < \varepsilon$. Deduce that the set of continuous functions of compact support is also dense in $L^p(\mathbb{R})$ for all $p \in [1, \infty)$.

**12.** Define a function $f$ on $\mathbb{R}$ by setting $f(x) = \exp(-1/x)$ for $x > 0$ and $f(x) = 0$ otherwise. Show that $f$ is $C^\infty$. Now let $\phi(x) = f(x)f(1-x)$ and $\psi(x) = c^{-1}\int_{-\infty}^{x}\phi(t)dt$, where $c = \int_{\mathbb{R}}\phi(t)dt$. Check that $\psi$ is $C^\infty$, $\psi(x) = 0$ if $x \leq 0$, $\psi(x) = 1$ if $x \geq 1$ and $\psi$ is non-decreasing. Use $\psi$ to build, for each interval $I = [a, b]$ and $\epsilon > 0$ a $C^\infty$-function $\psi_{I,\epsilon}$ on $\mathbb{R}$ such that

$$1_{[a,b]} \leq \psi_{I,\epsilon} \leq 1_{[a-\epsilon,b+\epsilon]}.$$

Use this to construct for each compact set $K \subset \mathbb{R}^d$ and each open set $U \supset K$ a $C^\infty$-function $\psi_{K,U}$ on $\mathbb{R}^d$ such that

$$1_K \leq \psi_{K,U} \leq 1_U.$$

Deduce that the smooth functions of compact support $C_c^\infty(\mathbb{R}^d)$ form a dense subspace of $L^p(\mathbb{R}^d)$ for any $p \in [1, +\infty)$.

**13.*.** Let $\mu$ be a Borel probability measure on $\mathbb{R}^d$. Show that there is a sequence of finitely supported probability measures $\mu_n$ on $\mathbb{R}^d$, which converges weakly to $\mu$.

**14.*.** Let $\mu$ and $\{\mu_n\}_{n\geq 1}$ be probability measures on $\mathbb{R}^d$. Show that the following are equivalent:

(1) $\mu_n$ converges weakly to $\mu$ as $n \to \infty$.

(2) there is a countable dense (for uniform convergence) sequence $(f_i)_{i\geq 1}$ in the space of continuous and bounded functions on $\mathbb{R}^d$, such that $\int f_i d\mu_n \to \int f_i d\mu$ as $n \to \infty$, for each $i$.

(3) for every Borel set $A$ such that $\mu(\partial A) = 0$ we have $\mu_n(A) \to \mu(A)$ as $n \to \infty$ (here $\partial A$ is the boundary of $A$, i.e. the points in the closure of $A$ that are not in the interior of $A$.)

**15.*.** [Existence of product measure on infinite products] Let $\{(\Omega_i, \mathcal{F}_i, \mu_i)\}_{i\geq 1}$ be a sequence of probability spaces. Let $\Omega = \prod_{i\geq 1}\Omega_i$. Let $\mathcal{C}$ be the Boolean algebra of *cylinder sets*, namely subsets of the form $B := A \times \prod_{i>n}\Omega_i$, where $A \subset \prod_{i=1}^{n}\Omega_i$ belongs to the product $\sigma$-algebra $\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$. Finally let $\mathcal{F} = \sigma(\mathcal{C})$. Show that there exists a unique probability measure $\mu$ on $(\Omega, \mathcal{F})$ such that

$$\mu(B) = \mu_1 \otimes \ldots \otimes \mu_n(A),$$

for every cylinder set $B = A \times \prod_{i>n}\Omega_i$ as above.

*Hint: check that $\mu$ is well-defined and finitely additive on $\mathcal{C}$. Then apply Caratheodory's extension theorem. The main point is to check $\sigma$-additivity: for this use the equivalent continuity axiom (ExSh 1, ex. 6). Given $B_{n+1} \subset B_n \in \mathcal{C}$ with $\mu(B_n) > \epsilon > 0$ for all $n$, show that there is an $\omega_1 \in \Omega_1$ whose slices $(B_n)_{\omega_1} := \{\omega = (\omega_i)_{i\geq 2} \in \prod_{i\geq 2}\Omega_i : (\omega_1, \omega) \in B_n\}$ have $\mu^{(1)}((B_n)_{\omega_1}) > \epsilon/2$ for all $n$, where $\mu^{(1)}$ is the projection of $\mu$ onto $\prod_{i\geq 2}\Omega_i$. Iterate and conclude that $\cap_n B_n \neq \varnothing$.*