E. Breuillard Michaelmas 2018

Probability and Measure 2

*Exercises marked with a star * are not examinable*

1. Let $(E, \mathcal{E}, \mu)$ be a measure space. Prove Scheffé's lemma : Let $(f_n : n \in \mathbb{N})$ be a sequence of integrable functions and suppose that $f_n \to f$ a.e. for some integrable function f. If $\mu(|f_n|) \to \mu(|f|)$, then $\mu(|f_n - f|) \to 0$.

2. Let $(E, \mathcal{E})$ and $(G, \mathcal{G})$ be measurable spaces and let $f : E \to G$ be a measurable function. Given a measure μ on $(E, \mathcal{E})$, the image measure $\nu := f_*\mu$ on $(G, \mathcal{G})$ is defined by

$$\nu(A) = \mu(f^{-1}(A)),$$

for all $A \in \mathcal{G}$. Show that ν is indeed a measure and that $\nu(g) = \mu(g \circ f)$ for all non-negative measurable functions g on G. In the case when $E = G = \mathbb{R}^d$ endowed with Lebesgue's measure m and $f \in GL_d(\mathbb{R})$ is an invertible linear map, show that $f_*m = \frac{1}{|\det f|}m$.

3. Let f be a real-valued integrable function on a measure space $(X, \mathcal{A}, \mu)$. Let $\mathcal{F}$ be a family of subsets from $\mathcal{A}$, which is stable under intersection, contains X and generates the σ-algebra $\mathcal{A}$. Suppose that $\mu(f 1_F) = 0$ for all subsets $F \in \mathcal{F}$. Show that $f = 0$ μ-a.e.

4. Let μ and ν be finite Borel measures on $\mathbb{R}$. Let f be a continuous bounded function on $\mathbb{R}$. Show that f is integrable with respect to μ and ν. Show further that, if $\mu(f) = \nu(f)$ for all such f, then $\mu = \nu$.

5. Show that the function $\sin x/x$ is not Lebesgue integrable over $[1,\infty)$ but that integral $\int_1^N (\sin x/x)dx$ converges as $N \to \infty$.

Show that the function $f(x) := x^2 \sin(\frac{1}{x^2})$ is continuous and differentiable at every point of $[0,1]$ but its derivative is not Lebesgue integrable on this interval.

6. Show that, as $n \to \infty$,

$$\int_0^\infty \sin(e^x)/(1+nx^2)dx \to 0 \quad \text{and} \quad \int_0^1 (n\cos x)/(1+n^2x^{\frac{3}{2}})dx \to 0.$$

7. Show that the product of the Borel σ-algebras of $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ is the Borel σ-algebra of $\mathbb{R}^{d_1+d_2}$. Give an example to show that this is no longer the case if the word Borel is replaced by Lebesgue.

8. Show that the following condition implies that random variables X and Y are independent: $\mathbb{P}(X \le x, Y \le y) = \mathbb{P}(X \le x)\mathbb{P}(Y \le y)$ for all $x, y \in \mathbb{R}$.

9. Let $(A_n : n \in \mathbb{N})$ be a sequence of events, with $\mathbb{P}(A_n) = 1/n^2$ for all n. Set $X_n = n^2 1_{A_n} - 1$ and set $\bar{X}_n = (X_1 + \cdots + X_n)/n$. Show that $\mathbb{E}(\bar{X}_n) = 0$ for all n, but that $\bar{X}_n \to -1$ almost surely as $n \to \infty$.

10. The zeta function is defined for $s > 1$ by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. Let X and Y be independent integer valued random variables with

$$\mathbb{P}(X = n) = \mathbb{P}(Y = n) = n^{-s}/\zeta(s).$$

Write A_n for the event that n divides X. Show that the events $(A_p : p \text{ prime})$ are independent and deduce Euler's formula

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right).$$

Show also that $\mathbb{P}(X \text{ is square-free}) = 1/\zeta(2s)$. Write H for the highest common factor of X and Y. Show finally that $\mathbb{P}(H = n) = n^{-2s}/\zeta(2s)$.

11. Let μ and ν be probability measures on $(E, \mathcal{E})$ and let $f : E \to [0, R]$ be a measurable function. Suppose that $\nu(A) = \mu(f 1_A)$ for all $A \in \mathcal{E}$. Let $(X_n : n \in \mathbb{N})$ be a sequence of i.i.d. random variables in E with law μ and let $(U_n : n \in \mathbb{N})$ be an independent sequence of i.i.d. random variables with uniform law in $[0, 1]$. Set

$$T = \min\{n \in \mathbb{N} : RU_n \leq f(X_n)\}, \text{ and } Y = X_T.$$

Show that Y has law ν. (This justifies simulation by rejection sampling.)

12.*. Let X be a second countable locally compact topological space (if you do not know what this means, assume $X = \mathbb{R}^d$). Let μ be a Radon measure on X (i.e. a measure on the Borel σ-algebra $\mathcal{B}(X)$ of X, which gives finite measure to every compact subset). Show that for every Borel subset $E \subset X$ with $\mu(E)$ finite, and for every $\epsilon > 0$ there is a compact subset K and an open subset U such that $K \subset E \subset U \subset X$ and

$$\mu(U/K) \leq \epsilon. \qquad (*)$$

Deduce that μ is *regular*, i.e. for every Borel subset E,

$$\mu(E) = \sup\{\mu(K); E \supset K \text{ compact }\} = \inf\{\mu(U); E \subset U \text{ open }\}.$$

Let $\mathcal{L}_\mu$ be the completion of $\mathcal{B}(X)$ with respect to μ. Show further that a subset $E \subset X$ is $\mathcal{L}_\mu$-measurable if and only if for every $\epsilon > 0$ there is a compact subset K and an open subset U of X such that $K \subset E \subset U$ and $\mu(U/E) < \epsilon$.

14.*. Recall that a bounded function $f : [0,1] \to \mathbb{R}$ is called Riemann integrable if all its Riemann sums converge. Let $\mathcal{P}_n$ be the level-n dyadic partition of $[0,1)$ given by all intervals of the form $I_{k,n} = [\frac{k}{2^n}, \frac{k+1}{2^n})$ for $k = 0, \ldots, 2^n - 1$, and let g_n be the step function equal to $\inf_{I_{k,n}} f$ on $I_{k,n}$ and f_n the step function equal to $\sup_{I_{k,n}} f$ on $I_{k,n}$.

Show that f is Riemann integrable if and only if $\int_{[0,1]} f_n - \int_{[0,1]} g_n$ tends to 0 as n tends to infinity (where the measure on $[0,1]$ is Lebesgue measure).

Let $\mathcal{D}$ be the set of all dyadic numbers (i.e. numbers of the form $\frac{k}{2^n}$ for some k, n). Show that if $x \in [0,1] \setminus \mathcal{D}$, then f is continuous at x if and only if $\lim_{n \to +\infty} f_n(x) - g_n(x) = 0$.

Deduce that f is Riemann integrable if and only if the set of discontinuity of f is of Lebesgue measure zero.

15.*. Give an example of a homeomorphism ϕ of $[0,1]$ and a Lebesgue measurable subset $E \subset [0,1]$ such that $\phi^{-1}(E)$ is not Lebesgue measurable. [*Hint: use a devil's staircase construction similar to Exercise 12 in Example Sheet 1, together with Vitali's counter-example*]