

Probability and Measure 4

*You are not required to do the exercises marked with a star **

1. Let μ be a Borel probability measure on \mathbb{R}^d and let $(\mu_n : n \in \mathbb{N})$ be a sequence of such measures. Suppose that $\mu_n(f) \rightarrow \mu(f)$ for all C^∞ functions on \mathbb{R}^d of compact support. Show that μ_n converges weakly to μ on \mathbb{R}^d .

2. Define a function ψ on \mathbb{R} by setting $\psi(x) = C \exp\{-(1-x^2)^{-1}\}$ for $|x| < 1$ and $\psi(x) = 0$ otherwise, where C is a constant chosen so that $\int_{\mathbb{R}} \psi(x) dx = 1$. For $f \in L^1(\mathbb{R})$ of compact support, show that $f * \psi$ is C^∞ and of compact support.

3. Let (E, \mathcal{E}, μ) be a measure space and let $V_1 \leq V_2 \leq \dots$ be an increasing sequence of closed subspaces of $L^2 = L^2(E, \mathcal{E}, \mu)$ for $f \in L^2$, denote by f_n the orthogonal projection of f on V_n . Show that f_n converges in L^2 .

4. Show that the Fourier transform of a finite Borel measure on \mathbb{R}^d is a bounded continuous function.

5*. Show that if f, g are two bounded integrable functions on \mathbb{R}^d then $f * g$ is continuous. Deduce that if $A \subset \mathbb{R}^d$ has positive Lebesgue measure, then $A - A := \{a - b; a, b \in A\}$ has non empty interior.

6. For a finite Borel measure μ on the line show that, if $\int |x|^k d\mu(x) < \infty$, then the Fourier transform $\hat{\mu}$ of μ has a k th continuous derivative, which at 0 is given by

$$\hat{\mu}^{(k)}(0) = i^k \int x^k d\mu(x).$$

7. Let X, Y be two bounded random variables such that $\mathbb{E}(X^k) = \mathbb{E}(Y^k)$ for all integers $k \geq 0$. Show that X and Y have the same distribution.

8. (i) Show that for any real numbers a, b one has $\int_a^b e^{itx} dx \rightarrow 0$ as $|t| \rightarrow \infty$.

(ii) Show that, for any $f \in L^1(\mathbb{R})$, the Fourier transform

$$\hat{f}(t) = \int_{-\infty}^{\infty} e^{itx} f(x) dx$$

tends to 0 as $|t| \rightarrow \infty$. This is the *Riemann–Lebesgue Lemma*.

9*. Say that $f \in L^2(\mathbb{R})$ is L^2 -differentiable if there is a function Df in $L^2(\mathbb{R})$ (called the L^2 -derivative of f) such that

$$\|\tau_h f - f - hDf\|_2/h \rightarrow 0 \quad \text{as } h \rightarrow 0,$$

where $\tau_h f(x) = f(x+h)$. For example show that the function $f(x) = \max(1 - |x|, 0)$ is L^2 -differentiable and find its L^2 -derivative.

Suppose that $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is L^2 -differentiable. Show that f has admits a continuous version (i.e. $\exists g$ continuous s.t. $g(x) = f(x)$ a.e.).

Hint: show that $u\hat{f}(u) \in L^2$ and that $\|f\|_{\infty} \leq C\|(1+|u|)\hat{f}(u)\|_2$ for some finite absolute constant C , to be determined. This is the Sobolev embedding theorem and Sobolev inequality.

10*. The Schwartz space \mathcal{S} of \mathbb{R} is the space of all C^{∞} complex valued functions f on \mathbb{R} such that for every $k, n \in \mathbb{Z}_{\geq 0}$ we have $f^{(k)}(x) = O(1/|x|^n)$ as $|x| \rightarrow +\infty$. Show that if f belongs to \mathcal{S} so does \hat{f} .

11. Let $X = (X_1, \dots, X_n)$ be a Gaussian random variable in \mathbb{R}^n with mean μ and covariance matrix V . Assume that V is invertible write $V^{-1/2}$ for the positive-definite square root of V^{-1} . Set $Y = (Y_1, \dots, Y_n) = V^{-1/2}(X - \mu)$. Show that Y_1, \dots, Y_n are independent $N(0, 1)$ random variables. Show further that we can write X_2 in the form $X_2 = aX_1 + Z$ where Z is independent of X_1 and determine the distribution of Z .

12*. Let X_1, \dots, X_n be independent $N(0, 1)$ random variables. Show that

$$\left(\bar{X}, \sum_{m=1}^n (X_m - \bar{X})^2 \right) \quad \text{and} \quad \left(\frac{X_n}{\sqrt{n}}, \sum_{m=1}^{n-1} X_m^2 \right)$$

have the same distribution, where $\bar{X} = (X_1 + \dots + X_n)/n$.

13. The Cauchy distribution has density function $f(x) = \pi^{-1}(1 + x^2)^{-1}$ for $x \in \mathbb{R}$. Show that one can simulate a random variable X whose law follows the Cauchy distribution by picking a random angle θ uniformly in $(-\frac{\pi}{2}, \frac{\pi}{2})$ and setting $X = \tan \theta$. Show that the corresponding characteristic function is given by $\varphi(u) = e^{-|u|}$. Show also that, if X_1, \dots, X_n are independent Cauchy random variables, then the random variable $(X_1 + \dots + X_n)/n$ is also Cauchy.

14*. For each $n \in \mathbb{N}$, there is a unique probability measure μ_n on the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ such that $\mu_n(A) = \mu_n(UA)$ for all Borel sets A and all orthogonal $n \times n$ matrices U . Fix $k \in \mathbb{N}$ and, for $n \geq k$, let γ_n denote the probability measure on \mathbb{R}^k which is the law of $\sqrt{n}(x^1, \dots, x^k)$ under μ_n . Show

(a) if $X \sim N(0, I_n)$ then $X/|X| \sim \mu_n$,

(b) if $(X_n : n \in \mathbb{N})$ is a sequence of independent $N(0, 1)$ random variables and if $R_n = \sqrt{X_1^2 + \dots + X_n^2}$ then $R_n/\sqrt{n} \rightarrow 1$ a.s.,

(c) γ_n converges weakly to the standard Gaussian distribution on \mathbb{R}^k as $n \rightarrow \infty$.

15. Let (E, \mathcal{E}, μ) be a measure space and $\theta : E \rightarrow E$ a measure-preserving transformation. Show that $\mathcal{E}_\theta := \{A \in \mathcal{E} : \theta^{-1}(A) = A\}$ is a σ -algebra, and that a measurable function f is \mathcal{E}_θ -measurable if and only if it is *invariant*, that is $f \circ \theta = f$.

16. Show that, if θ is an ergodic measure-preserving transformation and f is a θ -invariant function, then there exists a constant $c \in \mathbb{R}$ such that $f = c$ a.e..

17. For $x \in [0, 1)$, set $\theta(x) = 2x \bmod 1$. Show that θ is a measure-preserving transformation of $([0, 1), \mathcal{B}([0, 1)), dx)$, and that θ is ergodic. Identify the invariant function \bar{f} corresponding to each integrable function f .

18. Fix $a \in [0, 1)$ and define, for $x \in [0, 1)$, $\theta(x) = x + a \bmod 1$. Show that θ is also a measure-preserving transformation of $([0, 1), \mathcal{B}([0, 1)), dx)$. Determine for which values of a the transformation θ is ergodic. *Hint: you may use the fact that any integrable function f on $[0, 1)$ whose Fourier coefficients all vanish must itself vanish a.e..* Identify, for all values of a , the invariant function \bar{f} corresponding to an integrable function f .

19*. Call a sequence of random variables $(X_n : n \in \mathbb{N})$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ *stationary* if for each $n, k \in \mathbb{N}$ the random vectors (X_1, \dots, X_n) and $(X_{k+1}, \dots, X_{k+n})$ have the same distribution: for $A_1, \dots, A_n \in \mathcal{B}$,

$$\mathbb{P}(X_1 \in A_1, \dots, X_n \in A_n) = \mathbb{P}(X_{k+1} \in A_1, \dots, X_{k+n} \in A_n).$$

Show that, if $(X_n : n \in \mathbb{N})$ is a stationary sequence and $X_1 \in L^p$, for some $p \in [1, \infty)$, then

$$\frac{1}{n} \sum_{i=1}^n X_i \rightarrow X \quad \text{a.s. and in } L^p,$$

for some random variable $X \in L^p$ and find $\mathbb{E}(X)$.