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Probability and Measure 2

3.1. Let $(X, \mathcal{A})$ be a measurable space. Suppose that a simple function f has two representations

$$f = \sum_{k=1}^{m} a_k 1_{A_k} = \sum_{j=1}^{n} b_j 1_{B_j}.$$

Show that, for any measure μ,

$$\sum_{k=1}^{m} a_k \mu(A_k) = \sum_{j=1}^{n} b_j \mu(B_j).$$

hint: for $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m) \in \{0,1\}^m$, define $A_\varepsilon = A_1^{\varepsilon_1} \cap \ldots \cap A_m^{\varepsilon_m}$ where $A_k^0 = A_k^c$ and $A_k^1 = A_k$. Define similarly B_δ for $\delta \in \{0,1\}^n$. Then set $f_{\varepsilon,\delta} = \sum_{k=1}^{m} \varepsilon_k a_k$ if $A_\varepsilon \cap B_\delta \neq \emptyset$ and $f_{\varepsilon,\delta} = 0$ otherwise. Show then that

$$\sum_{k=1}^{m} a_k \mu(A_k) = \sum_{\varepsilon,\delta} f_{\varepsilon,\delta} \mu(A_\varepsilon \cap B_\delta)$$

3.2. Let μ and ν be finite Borel measures on $\mathbb{R}$. Let f be a continuous bounded function on $\mathbb{R}$. Show that f is integrable with respect to μ and ν. Show further that, if $\mu(f) = \nu(f)$ for all such f, then $\mu = \nu$.

3.3. Let f be a real-valued integrable function on a measure space $(X, \mathcal{A}, \mu)$. Let $\mathcal{F}$ be a family of subsets from $\mathcal{A}$, which is stable under intersection, contains X and generates the σ-algebra $\mathcal{A}$. Suppose that $\mu(f 1_F) = 0$ for all subsets $F \in \mathcal{F}$. Show that $f = 0$ μ-a.e.

3.4. Let X be a non-negative integer-valued random variable. Show that

$$\mathbb{E}(X) = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n).$$

Deduce that, if $\mathbb{E}(X) = \infty$ and $X_1, X_2, \ldots$ is a sequence of independent random variables with the same distribution as X, then, almost surely, $\limsup_n (X_n/n) = \infty$. (hint: use the second Borel-Cantelli lemma)

Now suppose that $Y_1, Y_2, \ldots$ is any sequence of independent identically distributed real-valued random variables with $\mathbb{E}|Y_1| = \infty$. Show that, almost surely, $\limsup_n (|Y_n|/n) = \infty$, and moreover $\limsup_n (|Y_1 + \cdots + Y_n|/n) = \infty$.

3.5. Show that the product of the Borel σ-algebras of $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ is the Borel σ-algebra of $\mathbb{R}^{d_1+d_2}$. Give an example to show that this is no longer the case if the word Borel is replaced by Lebesgue.

3.6. Show that the function $\sin x/x$ is not Lebesgue integrable over $[1, \infty)$ but that integral $\int_1^N (\sin x/x) dx$ converges as $N \to \infty$.

Show that the function $f(x) := x^2 \sin(\frac{1}{x^2})$ is continuous and differentiable at every point of $[0, 1]$ but its derivative is not Lebesgue integrable on this interval.

3.7. Show that, as $n \to \infty$,

$$\int_0^{\infty} \sin(e^x)/(1 + nx^2) dx \to 0 \quad \text{and} \quad \int_0^1 (n \cos x)/(1 + n^2 x^{\frac{3}{2}}) dx \to 0.$$

3.8. Let u and v be differentiable functions on $\mathbb{R}$ with continuous derivatives u' and v'. Suppose that uv' and $u'v$ are integrable on $\mathbb{R}$ and $u(x)v(x) \to 0$ as $|x| \to \infty$. Show that

$$\int_{\mathbb{R}} u(x)v'(x) dx = -\int_{\mathbb{R}} u'(x)v(x) dx.$$

3.9. Let $(E, \mathcal{E})$ and $(G, \mathcal{G})$ be measurable spaces and let $f : E \to G$ be a measurable function. Given a measure μ on $(E, \mathcal{E})$, the image measure $\nu := f_*\mu$ on $(G, \mathcal{G})$ is defined by

$$\nu(A) = \mu(f^{-1}(A)),$$

for all $A \in \mathcal{G}$. Show that $\nu(g) = \mu(g \circ f)$ for all non-negative measurable functions g on G.

3.10. The moment generating function ϕ of a real-valued random variable X is defined by $\phi(\theta) = \mathbb{E}(e^{\theta X}), \forall \theta \in \mathbb{R}$.

Suppose that ϕ is finite on an open interval containing 0. Show that ϕ has derivatives of all orders at 0 and that X has finite moments of all orders given by

$$\mathbb{E}(X^n) = \left(\frac{d}{d\theta}\right)^n \bigg|_{\theta=0} \phi(\theta).$$

3.11. Let $X_1, \ldots, X_n$ be random variables with density functions $f_1, \ldots, f_n$ respectively. Suppose that the $\mathbb{R}^n$-valued random variable $X = (X_1, \ldots, X_n)$ also has a density function f. Show that $X_1, \ldots, X_n$ are independent if and only if

$$f(x_1, \ldots, x_n) = f_1(x_1) \ldots f_n(x_n) \qquad \text{a.e.}$$

3.12. Recall that a bounded function $f : [0,1] \to \mathbb{R}$ is called Riemann integrable if all its Riemann sums converge. Let $\mathcal{P}_n$ be the level-n dyadic partition of $[0,1)$ given by all intervals of the form $I_{k,n} = [\frac{k}{2^n}, \frac{k+1}{2^n})$ for $k = 0, \ldots, 2^n - 1$, and let g_n be the step function equal to $\inf_{I_{k,n}} f$ on $I_{k,n}$ and f_n the step function equal to $\sup_{I_{k,n}} f$ on $I_{k,n}$.

Show that f is Riemann integrable if and only if $\int_{[0,1]} f_n - \int_{[0,1]} g_n$ tends to 0 as n tends to infinity (where the measure on $[0,1]$ is Lebesgue measure).

Let $\mathcal{D}$ be the set of all dyadic numbers (i.e. numbers of the form $\frac{k}{2^n}$ for some k, n). Show that if $x \in [0,1] \setminus \mathcal{D}$, then f is continuous at x if and only if $\lim_{n \to +\infty} f_n(x) - g_n(x) = 0$.

Deduce that f is Riemann integrable if and only if the set of discontinuity of f is of Lebesgue measure zero.

3.13. Let μ and ν be probability measures on $(E, \mathcal{E})$ and let $f : E \to [0, R]$ be a measurable function. Suppose that $\nu(A) = \mu(f 1_A)$ for all $A \in \mathcal{E}$. Let $(X_n : n \in \mathbb{N})$ be a sequence of independent random variables in E with law μ and let $(U_n : n \in \mathbb{N})$ be a sequence of independent uniformly distributed on $[0, R]$ random variables. Set

$$T = \min\{n \in \mathbb{N} : U_n \leq f(X_n)\}, Y = X_T.$$

Show that Y has law ν. (This justifies simulation by rejection sampling.)