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Probability and Measure 1

1.1. Let E be a set and let $\mathcal{S}$ be a set of σ-algebras on E. Define

$$\mathcal{E}^* = \{A \subseteq E : A \in \mathcal{E} \quad \text{for all} \quad \mathcal{E} \in \mathcal{S}\}.$$

Show that $\mathcal{E}^*$ is a σ-algebra on E. Show, on the other hand, by example, that the union of two σ-algebras on the same set need not be a σ-algebra.

1.2. Show that the following sets of subsets of $\mathbb{R}$ all generate the same σ-algebra:
(a) $\{(a,b) : a < b\}$, (b) $\{(a,b] : a < b\}$, (c) $\{(-\infty, b] : b \in \mathbb{R}\}$.

1.3. Let E be a set. Show that a countably additive set function on a Boolean algebra of subsets of E is additive, increasing and countably subadditive.

1.4. Let E be a set and $\mathcal{E}$ a family of subsets of E, which contains E and $\varnothing$, and is stable under complementation, under countable disjoint unions and under finite intersections. Show that $\mathcal{E}$ is a σ-algebra.

1.5. Let μ be a finite-valued additive set function on a Boolean algebra $\mathcal{A}$ of subsets of a set X. Show that μ is countably additive if and only if the following condition holds: for any decreasing sequence $(A_n : n \in \mathbb{N})$ of sets in $\mathcal{A}$, with $\cap_n A_n = \emptyset$, we have $\mu(A_n) \to 0$.

1.6. Let $(E, \mathcal{E}, \mu)$ be a finite measure space. Recall that for any sequence of sets $(A_n : n \in \mathbb{N})$ in $\mathcal{E}$, $\liminf A_n$ is the subset of those $x \in E$ such that $x \in A_m$ for all large enough $m \in \mathbb{N}$, and $\limsup A_n$ is the subset of those $x \in E$ such that x belongs to A_m for infinitely many $m \in \mathbb{N}$. Show that

$$\mu(\liminf A_n) \le \liminf \mu(A_n) \le \limsup \mu(A_n) \le \mu(\limsup A_n).$$

Show that the first inequality remains true without the assumption that $\mu(E) < \infty$, but that the last inequality may then be false.

1.7. A subset $E \subset \mathbb{R}$ is called Jordan measurable if for every $\epsilon > 0$ there are two finite unions of intervals $A = \bigcup_1^n I_i$ and $B = \bigcup_1^m J_j$ such that $A \subset E \subset B$ and $m(B \setminus A) < \varepsilon$, where m is defined on finite disjoint unions of intervals as the total length of the intervals.

Give an example of a compact subset of $[0, 1]$ that is not Jordan measurable.

1.8. Let $(E, \mathcal{E}, \mu)$ be a measure space. Call a subset $N \subseteq E$ *null* if $N \subseteq B$ for some $B \in \mathcal{E}$ with $\mu(B) = 0$. Write $\mathcal{N}$ for the set of null sets. Prove that the set of subsets $\mathcal{E}^\mu = \{A \cup N : A \in \mathcal{E}, N \in \mathcal{N}\}$ is a σ-algebra and show that μ has a well-defined and countably additive extension to $\mathcal{E}^\mu$ given by $\mu(A \cup N) = \mu(A)$. We call $\mathcal{E}^\mu$ the *completion of* $\mathcal{E}$ *with respect to* μ. Suppose now that E is σ-finite and write μ^* for the outer measure associated to μ, as in the proof of Carathéodory's Extension Theorem. Show that $\mathcal{E}^\mu$ is exactly the set of μ^*-measurable sets.

1.9. Recall that the outer measure $m^*(E)$ of a subset E of $\mathbb{R}^d$ is defined as

$$m^*(E) = \inf \sum_n m(B_n)$$

where the infinimum is taken over all covers of E by countable unions $\bigcup_{n \in \mathbb{N}} B_n$ of boxes $B_n \subset \mathbb{R}^d$, and $m(B_n)$ is the product of the side lengths of the box B_n.

Let E be a subset of $X := [0, 1]^d$. In Lebesgue's 1901 original article, E is defined to be (Lebesgue) measurable if $m^*(E) + m^*(X \setminus E) = 1$. Show that this definition equivalent to the one(s) given in class.

2.1. Let $(f_n : n \in \mathbb{N})$ be a sequence of measurable functions on a measurable space $(E, \mathcal{E})$. Show that the following functions are also measurable: $f_1 + f_2$, $f_1 f_2$, $\inf_n f_n$, $\sup_n f_n$, $\liminf_n f_n$, $\limsup_n f_n$. Show also that $\{x \in E : f_n(x) \text{ converges as } n \to \infty\} \in \mathcal{E}$.

2.2. Let $(E, \mathcal{E})$ and $(G, \mathcal{G})$ be measurable spaces, let μ be a measure on $\mathcal{E}$, and let $f : E \to G$ be a measurable function. Show that we can define a measure ν on $\mathcal{G}$ by setting $\nu(A) = \mu(f^{-1}(A))$ for each $A \in \mathcal{G}$.

2.3. Show that the following condition implies that random variables X and Y are independent: $\mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y)$ for all $x, y \in \mathbb{R}$.

2.4. Let $(A_n : n \in \mathbb{N})$ be a sequence of events in a probability space. Show that the events A_n are independent if and only if the σ-algebras $\sigma(A_n) = \{\emptyset, A_n, A_n^c, \Omega\}$ are independent.

2.5. Let $(A_n : n \in \mathbb{N})$ be a sequence of events, with $\mathbb{P}(A_n) = 1/n^2$ for all n. Set $X_n = n^2 1_{A_n} - 1$ and set $\bar{X}_n = (X_1 + \cdots + X_n)/n$. Show that $\mathbb{E}(\bar{X}_n) = 0$ for all n, but that $\bar{X}_n \to -1$ almost surely as $n \to \infty$.

2.6. The zeta function is defined for $s > 1$ by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. Let X and Y be independent integer valued random variables with

$$\mathbb{P}(X = n) = \mathbb{P}(Y = n) = n^{-s}/\zeta(s).$$

Write A_n for the event that n divides X. Show that the events $(A_p : p \text{ prime})$ are independent and deduce Euler's formula

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right).$$

Show also that $\mathbb{P}(X \text{ is square-free}) = 1/\zeta(2s)$. Write H for the highest common factor of X and Y. Show finally that $\mathbb{P}(H = n) = n^{-2s}/\zeta(2s)$.

2.7. Let $(X_n : n \in \mathbb{N})$ be independent $N(0,1)$ random variables. Prove that

$$\limsup_n \left(X_n/\sqrt{2\log n}\right) = 1 \quad \text{a.s.}$$

2.8. Let C_n denote the nth approximation to the Cantor set C: thus $C_0 = [0,1]$, $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, etc. and $C_n \downarrow C$ as $n \to \infty$. Denote by F_n the distribution function of a random variable uniformly distributed on C_n. Show that

(a) C is uncountable and has Lebesgue measure 0,
(b) for all $x \in [0,1]$, the limit $F(x) = \lim_{n\to\infty} F_n(x)$ exists,
(c) the function F is continuous on $[0,1]$, with $F(0) = 0$ and $F(1) = 1$,
(d) for almost all $x \in [0,1]$, F is differentiable at x with $F'(x) = 0$.

Hint: express F_{n+1} recursively in terms of F_n and use this relation to obtain a uniform estimate on $F_{n+1} - F_n$.