PRINCIPLES OF STATISTICS – EXAMPLES 3/4

Part II, Michaelmas 2022, Po-Ling Loh (email: pll28@cam.ac.uk)
Questions by courtesy of Richard Nickl

Throughout, for observations X arising from a parametric model $\{f(\cdot,\theta) : \theta \in \Theta\}, \Theta \subseteq \mathbb{R}$, the quadratic risk of a decision rule $\delta(X)$ is defined to be $R(\delta,\theta) = E_\theta(\delta(X) - \theta)^2$.

1. Consider $X|\theta \sim Poisson(\theta)$, where $\theta \in \Theta = [0,\infty)$, and suppose the prior for θ is a Gamma distribution with parameters (α,λ). Show that the posterior distribution $\theta|X$ is also a Gamma distribution and find its parameters.

2. For $n \in \mathbb{N}$ fixed, suppose X is binomially $Bin(n,\theta)$-distributed, where $\theta \in \Theta = [0,1]$.

(a) Consider a prior for θ from a Beta(a,b) distribution, where $a,b > 0$. Show that the posterior distribution is Beta$(a+X, b+n-X)$, and compute the posterior mean $\bar{\theta}_n(X) = E(\theta|X)$.

(b) Show that the maximum likelihood estimator for θ is *not* identical to the posterior mean with "ignorant" uniform prior $\theta \sim U[0,1]$.

(c) Assuming that X is sampled from a fixed $Bin(n,\theta_0)$ distribution with $\theta_0 \in (0,1)$, derive the asymptotic distribution of $\sqrt{n}(\bar{\theta}_n(X) - \theta_0)$ as $n \to \infty$.

3. Let $X_1,\ldots,X_n$ be i.i.d. copies of a random variable X, and consider the Bayesian model $X|\theta \sim N(\theta,1)$ with prior π as $\theta \sim N(\mu,v^2)$. For $0 < \alpha < 1$, consider the credible interval

$$C_n = \{\theta \in \mathbb{R} : |\theta - E^\pi(\theta|X_1,\ldots,X_n)| \le R_n\},$$

where R_n is chosen such that $\pi(C_n|X_1,\ldots,X_n) = 1-\alpha$. Now assume $X \sim N(\theta_0,1)$ for some fixed $\theta_0 \in \mathbb{R}$, and show that, as $n \to \infty$, $P_{\theta_0}^{\mathbb{N}}(\theta_0 \in C_n) \to 1-\alpha$.

4. In a general decision problem, show that (a) a decision rule δ that has constant risk and is admissible is also minimax, and (b) any unique Bayes rule is admissible.

5. Consider an observation X from a parametric model $\{f(\cdot,\theta) : \theta \in \Theta\}$ with prior π on $\Theta \subseteq \mathbb{R}$ and a general risk function $R(\delta,\theta) = E_\theta L(\delta(X),\theta)$. Assume that there exists some decision rule δ_0 such that $R(\delta_0,\theta) < \infty$ for all $\theta \in \Theta$.

(a) For the loss function $L(a,\theta) = |a-\theta|$, show that the Bayes rule associated with π equals any median of the posterior distribution $\pi(\cdot|X)$.

(b) For weight function $w : \Theta \to [0,\infty)$ and loss function $L(a,\theta) = w(\theta)[a-\theta]^2$, show that the Bayes rule δ_π associated with π is unique and equals

$$\delta_\pi(X) = \frac{E^\pi[w(\theta)\theta|X]}{E^\pi[w(\theta)|X]},$$

assuming that the expectations in the last ratio exist and are finite, and $E^\pi[w(\theta)|X] > 0$.

6. (a) Considering $X_1,\ldots,X_n$ i.i.d. from a $N(\theta,1)$-model with $\theta \in \Theta = \mathbb{R}$, show that the maximum likelihood estimator is *not* a Bayes rule for estimating θ in quadratic risk for any prior distribution π.

(b) Let $X \sim Bin(n,\theta)$, where $\theta \in \Theta = [0,1]$. Find all prior distributions π on Θ for which the maximum likelihood estimator is a Bayes rule for estimating θ in quadratic risk.

7. Consider estimating $\theta \in \Theta = [0,1]$ in a $Bin(n,\theta)$ model under the quadratic risk.

(a) Find the unique minimax estimator $\tilde{\theta}_n$ of θ, and deduce that the maximum likelihood estimator $\hat{\theta}_n$ of θ is *not* minimax for a fixed sample size $n \in \mathbb{N}$. [Hint: Find first a Bayes rule with constant risk in $\theta \in \Theta$.]

(b) Show, however, that the maximum likelihood estimator dominates $\tilde{\theta}_n$ in the large sample limit by proving that, as $n \to \infty$,

$$\lim_n \frac{\sup_\theta R(\hat{\theta}_n, \theta)}{\sup_\theta R(\tilde{\theta}_n, \theta)} = 1$$

and

$$\lim_n \frac{R(\hat{\theta}_n, \theta)}{R(\tilde{\theta}_n, \theta)} < 1 \text{ for all } \theta \in [0,1], \theta \neq \frac{1}{2}.$$

8. Consider $X_1, \dots, X_n$ i.i.d. from a $N(\theta, 1)$ model, where $\theta \in \Theta = [0, \infty)$. Show that the sample mean $\bar{X}_n$ is inadmissible for quadratic risk, but that it is still minimax. What happens if $\Theta = [a, b]$ for some $0 < a < b < \infty$?

9. Let X be multivariate normal $N(\theta, I)$, where $\theta \in \Theta = \mathbb{R}^p, p \geq 3$, and I is the $p \times p$ identity matrix. Consider estimators

$$\tilde{\theta}^{(c)} = \left(1 - c\frac{p-2}{\|X\|^2}\right) X, \; 0 < c < 2,$$

for θ, under the risk function $R(\delta, \theta) = E_\theta \|\delta(X) - \theta\|^2$, where $\| \cdot \|$ is the standard Euclidean norm on $\mathbb{R}^p$.

(a) Show that the James-Stein estimator $\tilde{\theta}^{(1)}$ dominates all estimators $\tilde{\theta}^{(c)}, c \neq 1$.

(b) Let $\hat{\theta}$ be the maximum likelihood estimator of θ. Show that, for any $0 < c < 2$,

$$\sup_{\theta \in \Theta} R(\tilde{\theta}^{(c)}, \theta) = \sup_{\theta \in \Theta} R(\hat{\theta}, \theta).$$

10. Consider $X_1, \dots, X_n$ i.i.d. from a $N(\theta, 1)$ model with $\theta \in \Theta = \mathbb{R}$, and recall the Hodges' estimator

$$\tilde{\theta}_n = \bar{X}_n 1\{|\bar{X}_n| \geq n^{-1/4}\},$$

equal to the maximum likelihood estimator $\bar{X}_n$ of θ whenever $|\bar{X}_n| \geq n^{-1/4}$, and zero otherwise. Derive the asymptotic distribution of $\sqrt{n}(\tilde{\theta}_n - \theta)$ as $n \to \infty$ under P_θ for every $\theta \in \Theta$, and compare it to the asymptotic distribution of $\sqrt{n}(\bar{X}_n - \theta)$. Now compute the asymptotic maximal risk

$$\lim_n \sup_{\theta \in \Theta} E_\theta [\sqrt{n}(T_n - \theta)]^2$$

for both $T_n = \bar{X}_n$ and $T_n = \tilde{\theta}_n$.