II Number Theory – Example Sheet 3

Michaelmas 2024 jat58@cam.ac.uk

Throughout this sheet, log denotes logarithm to the base e, ϕ denotes the Euler totient function, μ the Möbius function, $\tau(n)$ the number of positive divisors of n, and $\sigma(n)$ the sum of the positive divisors of n.

1. Prove that for $\mathrm{Re}(s) > 1$, we have
$$\zeta(s)^2 = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}.$$
Can you find Dirichlet series for $1/\zeta(s)$ and $\zeta(s-1)/\zeta(s)$ (for suitable values of s)?

2. Find all natural numbers n for which $\sigma(n) + \phi(n) = n\tau(n)$.

3. Compute $\sum_{d|n} \Lambda(d)$ for natural numbers n. (Here Λ is the von Mangoldt function.)

4. Use Legendre's formula to compute $\pi(207)$.

5. Let N be a positive integer greater than 1.
   (i) Prove the inequality $N! > (\frac{N}{e})^N$.
   (ii) Deduce that
$$\sum_{p \le N} \frac{\log p}{p-1} > (\log N) - 1.$$

6. Prove that if $N > 1$ then $\sum_{n=1}^{N} \frac{1}{n}$ is not an integer.

7. Prove that for every $n \geq 1$, the set of numbers $\{1, 2, \ldots, 2n\}$ can be partitioned into pairs $\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_n, b_n\}$ so that the sum $a_i + b_i$ of each pair is prime.

8. (Harder) Prove that every integer $N > 6$ can be expressed as a sum of distinct primes.

9. Using a calculator, find $a_0, \ldots, a_4$ in the continued fraction expansions of e and π.

10. Let a be a positive integer. Determine explicitly the real number whose continued fraction is $[a, a, a, \ldots]$.

11. Determine the continued fraction expansions of $\sqrt{3}$, $\sqrt{7}$, $\sqrt{13}$, $\sqrt{19}$.

12. Let d be a positive integer that is not a square. Let θ_n and p_n/q_n be the complete quotients and convergents arising in the continued fraction expansion of $\sqrt{d}$. Show that for all $n \geq 1$ we have $p_{n-1} - q_{n-1}\sqrt{d} = (-1)^n / \prod_{i=1}^{n} \theta_i$.

13. (i) Let $(p_j/q_j)_{j=0,\ldots,n}$ be the convergents of the continued fraction $[a_0, \ldots, a_n]$. Show that the $(n-1)$th and nth convergents of $[a_n, \ldots, a_0]$ are q_n/q_{n-1} and p_n/p_{n-1}, respectively.
    (ii) Let $d > 1$ be a square-free integer, and let $\theta = r + s\sqrt{d}$ be a quadratic irrational with purely periodic continued fraction expansion $[\overline{a_1, \ldots, a_n}]$. Show that $-1/\theta' = [\overline{a_n, \ldots, a_1}]$, where $\theta' = r - s\sqrt{d}$ is the conjugate irrational.