Michaelmas Term 2023

Number Theory: Example Sheet 3 of 4

Throughout this sheet, ϕ denotes the Euler totient function, μ the Möbius function, $\tau(n)$ the number of positive divisors of n, and $\sigma(n)$ the sum of the positive divisors of n.

1. Prove that for $\mathrm{Re}(s) > 1$, we have
$$\zeta(s)^2 = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}.$$
Can you find Dirichlet series for $1/\zeta(s)$ and $\zeta(s-1)/\zeta(s)$ (for suitable values of s)?

2. Find all natural numbers n for which $\sigma(n) + \phi(n) = n\tau(n)$.

3. (i) Define the Möbius function μ, and check that it is multiplicative.
   (ii) Let f be a function defined on the natural numbers, and define g by $g(n) = \sum_{d|n} \mu(d) f(\frac{n}{d})$. Find an expression for f in terms of g.
   (iii) Find a relationship between μ and ϕ.

4. Compute $\sum_{d|n} \Lambda(d)$ for natural numbers n. (Here Λ is the von Mangoldt function.)

5. Use Legendre's formula to compute $\pi(207)$.

6. Let N be a positive integer greater than 1.
   (i) Show that the exact power of a prime p dividing $N!$ is $\sum_{k=1}^{\infty} \lfloor \frac{N}{p^k} \rfloor$.
   (ii) Prove the inequality $N! > (\frac{N}{e})^N$.
   (iii) Deduce that
$$\sum_{p \leqslant N} \frac{\log p}{p-1} > (\log N) - 1.$$

7. Prove that every non-constant polynomial with integer coefficients assumes infinitely many composite values.

8. Prove that every integer $N > 6$ can be expressed as a sum of distinct primes. (One method is to find a strictly increasing sequence of integers (a_k) such that every integer $6 < N \leqslant a_k$ is a sum of distinct primes less than or equal to the kth prime.)

9. Prove that for every $n \geqslant 1$, the set of numbers $\{1, 2, \ldots, 2n\}$ can be partitioned into pairs $\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_n, b_n\}$ so that the sum $a_i + b_i$ of each pair is prime.

10. Calculate $a_0, \ldots, a_4$ in the continued fraction expansions of e and π.

11. Let a be a positive integer. Determine explicitly the real number whose continued fraction is $[a, a, a, \ldots]$.

12. Determine the continued fraction expansions of $\sqrt{3}$, $\sqrt{7}$, $\sqrt{13}$, $\sqrt{19}$.

13. Let d be a positive integer that is not a square. Let θ_n and p_n/q_n be the complete quotients and convergents arising in the continued fraction expansion of $\sqrt{d}$. Show that for all $n \geqslant 1$ we have $p_{n-1} - q_{n-1}\sqrt{d} = (-1)^n / \prod_{i=1}^{n} \theta_i$.

14. (Extra question, requires IB Analysis & Topology.) Let χ_4 be the non-trivial group homomorphism $(\mathbb{Z}/4\mathbb{Z})^\times \to \{\pm 1\}$. Show that

$$L(s,\chi_4) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \frac{1}{9^s} - \frac{1}{11^s} + \dots$$

is a continuous function on $(0,\infty)$ with $L(1,\chi_4) \neq 0$. Use the Euler products to show that for $s > 1$ we have

$$\begin{aligned} \log \zeta(s) &= \sum_p \frac{1}{p^s} + g_1(s) \\ \log L(s,\chi_4) &= \sum_{p \neq 2} \frac{\chi_4(p)}{p^s} + g_2(s) \end{aligned}$$

where g_1 and g_2 are bounded functions. Deduce a special case of Dirichlet's theorem on primes in arithmetic progression.