Michaelmas Term 2023

Number Theory: Example Sheet 2 of 4

1. Evaluate the following Jacobi symbols (in fact, they are Legendre symbols):
$$\left(\frac{20964}{1987}\right), \qquad \left(\frac{741}{9283}\right), \qquad \left(\frac{5}{160465489}\right), \qquad \left(\frac{3083}{3911}\right).$$
Did it help to know that they are Legendre symbols?

2. Find all odd primes p for which 21 is a quadratic residue modulo p.

3. Prove that 3 is a quadratic non-residue modulo any Mersenne prime $2^n - 1$, with $n > 2$.

4. Let p be a prime with $p \equiv 1 \pmod 4$. Prove that the sum of the quadratic residues in the interval $[1, p-1]$ is equal to the sum of the quadratic non-residues in this interval. Does this hold if $p \equiv 3 \pmod 4$?

5. Let p be an odd prime and $\zeta = e^{2\pi i/p}$. Let
$$\tau = \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) \zeta^a.$$
Show that $\tau^2 = \pm p$, and determine how the sign depends on p.

6. Let a be a positive integer that is not a square. Prove that there are infinitely many odd primes p such that $\left(\frac{a}{p}\right) = -1$.

7. Let p be a prime with $p \equiv 3 \pmod 8$. Show that
$$\sum_{a=1}^{p-1} a\left(\frac{a}{p}\right) = \sum_{a=1}^{(p-1)/2} (2a-p)\left(\frac{a}{p}\right)$$
and
$$\sum_{a=1}^{p-1} a\left(\frac{a}{p}\right) = \sum_{a=1}^{(p-1)/2} (p-4a)\left(\frac{a}{p}\right).$$
Deduce that if $p > 3$ then
$$\sum_{a=1}^{(p-1)/2} \left(\frac{a}{p}\right) \equiv 0 \pmod 3.$$

8. Are the forms $3x^2 + 2xy + 23y^2$ and $2x^2 + 4xy + 5y^2$ equivalent (under the action of $\mathrm{SL}_2(\mathbb{Z})$)? Are the forms $15x^2 - 15xy + 4y^2$ and $3x^2 + 9xy + 8y^2$ equivalent?

9. Prove that equivalence of binary quadratic forms is an equivalence relation.

10. Make a list of all reduced positive definite binary quadratic forms of discriminant $-d$, where $d = 8, 11, 12, 16, 19, 23, 163$.

11. Find the smallest positive integer that can be represented by the form $4x^2 + 17xy + 20y^2$. What is the next largest? And the next?

12. Find congruence conditions for a prime p to be represented by the form $x^2 + 3y^2$.

13. Is there a positive definite binary quadratic form that represents 2 and the primes congruent to 1 or 3 modulo 8, but no other primes? Is there such a form representing the primes congruent to 1 modulo 4 only?

14. Find congruence conditions for a positive integer n to be properly represented by at least one of the two forms $x^2 + xy + 4y^2$ and $2x^2 + xy + 2y^2$.

    Assume that n is coprime to 15, and properly represented by at least one of the forms. Show that congruence conditions modulo 15 allow one to decide which form represents n.

15. (Optional, for enthusiasts of group theory) Let G be a group, and H a subgroup of finite index $(G : H) = m$. Write G as the union of cosets $x_i H$ $(1 \le i \le m)$. Then if $g \in G$, $gx_iH = x_jH$ for some j (depending on g). Write $\bar{g}$ for the mapping $i \mapsto j$. Recall (proof of Cayley's Theorem) that $g \mapsto \bar{g}$ is a permutation of $\{1, \ldots, m\}$ and that the map $g \mapsto \bar{g}$ is a homomorphism from G to the symmetric group S_m.

    From now on, assume that H is abelian.

    (i) Show that for every $g \in G$, the product

    $$T(g) = \prod_{i=1}^{m} x_{\bar{g}(i)}^{-1} g x_i$$

    belongs to H, and that T is a homomorphism from G to H (called the *transfer* homomorphism).

    (ii) Show that if G is abelian, then $T(g) = g^m$.

    (iii) Explain why, when $G = (\mathbb{Z}/p\mathbb{Z})^*$ and $H = \{\pm 1\}$, (ii) is just Gauss's Lemma.