Number Theory: Example Sheet 3 of 4

Throughout this sheet, ϕ denotes the Euler totient function, μ the Möbius function, Λ the von Mangoldt function, ζ the Riemann zeta function, $\pi(n)$ the number of primes up to n, $\tau(n)$ the number of positive divisors of n, and $\sigma(n)$ the sum of the positive divisors of n.

1. Prove that for $\mathrm{Re}(s) > 1$, we have

$$\zeta(s)^2 = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}.$$

   Can you find Dirichlet series for $1/\zeta(s)$ and $\zeta(s-1)/\zeta(s)$ (for suitable values of s)?

2. Find all natural numbers n for which $\sigma(n) + \phi(n) = n\tau(n)$.

3. (i) Define the Möbius function μ, and check that it is multiplicative.
   (ii) Let f be a function defined on the natural numbers, and define g by $g(n) = \sum_{d|n} \mu(d) f(\frac{n}{d})$. Find an expression for f in terms of g.
   (iii) Find a relationship between μ and ϕ.

4. Compute $\sum_{d|n} \Lambda(d)$ for natural numbers n.

5. Use Legendre's formula to compute $\pi(207)$.

6. Let N be a positive integer greater than 1.
   (i) Show that the exact power of a prime p dividing $N!$ is $\sum_{k=1}^{\infty} \lfloor \frac{N}{p^k} \rfloor$.
   (ii) Prove the inequality $N! > (\frac{N}{e})^N$.
   (iii) Deduce that

$$\sum_{p \leqslant N} \frac{\log p}{p-1} > (\log N) - 1.$$

7. Let $\theta(x) = \sum_{p \leqslant x} \log p$ and $\psi(x) = \sum_{n \leqslant x} \Lambda(n)$. Prove that for any $0 < \delta < 1$ we have

$$\delta(\pi(x) - \pi(x^\delta)) \log x \leqslant \theta(x) \leqslant \pi(x) \log x.$$

   By also finding suitable bounds for $\psi(x)$, deduce that $\theta(x) \sim \psi(x) \sim \pi(x) \log x$ as $x \to \infty$.

8. By considering the binomial coefficient $\binom{2m+1}{m}$ show that

$$\prod_{m+2 \leqslant p \leqslant 2m+1} p \leqslant 4^m.$$

   Deduce that $\prod_{p \leqslant n} p \leqslant 4^n$ for all $n \geqslant 1$.

9. Prove that for every $n \geqslant 1$, the set of numbers $\{1, 2, \ldots, 2n\}$ can be partitioned into pairs $\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_n, b_n\}$ so that the sum $a_i + b_i$ of each pair is prime.

10. Calculate $a_0, \ldots, a_4$ in the continued fraction expansions of e and π.

11. Let a be a positive integer. Determine explicitly the real number whose continued fraction is $[a, a, a, \ldots]$.

12. Determine the continued fraction expansions of $\sqrt{3}$, $\sqrt{7}$, $\sqrt{13}$, $\sqrt{19}$.

13. Let d be a positive integer that is not a square. Let θ_n and p_n/q_n be the complete quotients and convergents arising in the continued fraction expansion of $\sqrt{d}$. Show that for all $n \geqslant 1$ we have $p_{n-1} - q_{n-1}\sqrt{d} = (-1)^n / \prod_{i=1}^{n} \theta_i$.

14. (Optional, for enthusiasts of analysis.) Let χ_4 be the non-trivial group homomorphism $(\mathbb{Z}/4\mathbb{Z})^\times \to \{\pm 1\}$. Show that

$$L(s, \chi_4) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \frac{1}{9^s} - \frac{1}{11^s} + \ldots$$

is a continuous function on $(0, \infty)$ with $L(1, \chi_4) \neq 0$. Use the Euler products to show that for $s > 1$ we have

$$\log \zeta(s) = \sum_p \frac{1}{p^s} + g_1(s)$$

$$\log L(s, \chi_4) = \sum_{p \neq 2} \frac{\chi_4(p)}{p^s} + g_2(s)$$

where g_1 and g_2 are bounded functions. Deduce a special case of Dirichlet's theorem on primes in arithmetic progression.