

### Number Theory: Example Sheet 3 of 4

Throughout this sheet,  $\phi$  denotes the Euler totient function,  $\mu$  the Möbius function,  $\tau(n)$  the number of positive divisors of  $n$ , and  $\sigma(n)$  the sum of the positive divisors of  $n$ .

1. Prove that for  $\operatorname{Re}(s) > 1$ , we have

$$\zeta(s)^2 = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}.$$

Can you find Dirichlet series for  $1/\zeta(s)$  and  $\zeta(s-1)/\zeta(s)$  (for suitable values of  $s$ )?

2. Find all natural numbers  $n$  for which  $\sigma(n) + \phi(n) = n\tau(n)$ .
3. (i) Define the Möbius function  $\mu$ , and check that it is multiplicative.  
(ii) Let  $f$  be a function defined on the natural numbers, and define  $g$  by  $g(n) = \sum_{d|n} \mu(d)f\left(\frac{n}{d}\right)$ . Find an expression for  $f$  in terms of  $g$ .  
(iii) Find a relationship between  $\mu$  and  $\phi$ .
4. Compute  $\sum_{d|n} \Lambda(d)$  for natural numbers  $n$ . (Here  $\Lambda$  is the von Mangoldt function.)
5. Use Legendre's formula to compute  $\pi(207)$ .
6. Let  $N$  be a positive integer greater than 1.
  - (i) Show that the exact power of a prime  $p$  dividing  $N!$  is  $\sum_{k=1}^{\infty} \left\lfloor \frac{N}{p^k} \right\rfloor$ .  
(ii) Prove the inequality  $N! > \left(\frac{N}{e}\right)^N$ .  
(iii) Deduce that
$$\sum_{p \leq N} \frac{\log p}{p-1} > (\log N) - 1.$$
7. Prove that every non-constant polynomial with integer coefficients assumes infinitely many composite values.
8. Prove that every integer  $N > 6$  can be expressed as a sum of distinct primes. (One method is to find a strictly increasing sequence of integers  $(a_k)$  such that every integer  $6 < N \leq a_k$  is a sum of distinct primes less than or equal to the  $k$ th prime.)
9. Prove that for every  $n \geq 1$ , the set of numbers  $\{1, 2, \dots, 2n\}$  can be partitioned into pairs  $\{a_1, b_1\}, \{a_2, b_2\}, \dots, \{a_n, b_n\}$  so that the sum  $a_i + b_i$  of each pair is prime.
10. Calculate  $a_0, \dots, a_4$  in the continued fraction expansions of  $e$  and  $\pi$ .
11. Let  $a$  be a positive integer. Determine explicitly the real number whose continued fraction is  $[a, a, a, \dots]$ .

12. Determine the continued fraction expansions of  $\sqrt{3}$ ,  $\sqrt{7}$ ,  $\sqrt{13}$ ,  $\sqrt{19}$ .

13. Let  $d$  be a positive integer that is not a square. Let  $\theta_n$  and  $p_n/q_n$  be the complete quotients and convergents arising in the continued fraction expansion of  $\sqrt{d}$ . Show that for all  $n \geq 1$  we have  $p_{n-1} - q_{n-1}\sqrt{d} = (-1)^n / \prod_{i=1}^n \theta_i$ .

14. (Extra question, requires Analysis II.) Let  $\chi_4$  be the non-trivial group homomorphism  $(\mathbb{Z}/4\mathbb{Z})^\times \rightarrow \{\pm 1\}$ . Show that

$$L(s, \chi_4) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \frac{1}{9^s} - \frac{1}{11^s} + \dots$$

is a continuous function on  $(0, \infty)$  with  $L(1, \chi_4) \neq 0$ . Use the Euler products to show that for  $s > 1$  we have

$$\begin{aligned} \log \zeta(s) &= \sum_p \frac{1}{p^s} + g_1(s) \\ \log L(s, \chi_4) &= \sum_{p \neq 2} \frac{\chi_4(p)}{p^s} + g_2(s) \end{aligned}$$

where  $g_1$  and  $g_2$  are bounded functions. Deduce a special case of Dirichlet's theorem on primes in arithmetic progression.