Number Theory: Example Sheet 2 of 4

1. Evaluate the following Jacobi symbols (in fact, they are Legendre symbols):

$$\left(\frac{20964}{1987}\right), \qquad \left(\frac{741}{9283}\right), \qquad \left(\frac{5}{160465489}\right), \qquad \left(\frac{3083}{3911}\right).$$

   Did it help to know that they are Legendre symbols?

2. Find all odd primes p for which 21 is a quadratic residue modulo p.

3. Prove that 3 is a quadratic non-residue modulo any Mersenne prime $2^n - 1$, with $n > 2$.

4. Let p be a prime with $p \equiv 1 \pmod 4$. Prove that the sum of the quadratic residues in the interval $[1, p-1]$ is equal to the sum of the quadratic non-residues in this interval. Does this hold if $p \equiv 3 \pmod 4$?

5. Let p be an odd prime and $\zeta = e^{2\pi i/p}$. Let

$$\tau = \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) \zeta^a.$$

   Show that $\tau^2 = \pm p$, and determine how the sign depends on p.

6. Let a be a positive integer that is not a square. Prove that there are infinitely many odd primes p such that $\left(\frac{a}{p}\right) = -1$.

7. Let p be a prime with $p \equiv 3 \pmod 8$. Show that

$$\sum_{a=1}^{p-1} a \left(\frac{a}{p}\right) = \sum_{a=1}^{(p-1)/2} (2a - p) \left(\frac{a}{p}\right)$$

   and

$$\sum_{a=1}^{p-1} a \left(\frac{a}{p}\right) = \sum_{a=1}^{(p-1)/2} (p - 4a) \left(\frac{a}{p}\right).$$

   Deduce that if $p > 3$ then

$$\sum_{a=1}^{(p-1)/2} \left(\frac{a}{p}\right) \equiv 0 \pmod 3.$$

8. Are the forms $3x^2 + 2xy + 23y^2$ and $2x^2 + 4xy + 5y^2$ equivalent (under the action of $\mathrm{SL}_2(\mathbb{Z})$)? Are the forms $15x^2 - 15xy + 4y^2$ and $3x^2 + 9xy + 8y^2$ equivalent?

9. Prove that equivalence of binary quadratic forms is an equivalence relation.
10. Make a list of all reduced positive definite quadratic forms of discriminant $-d$, where $d = 8$, 11, 12, 16, 19, 23, 163.
11. Find the smallest positive integer that can be represented by the form $4x^2+17xy+20y^2$. What is the next largest? And the next?
12. Find congruence conditions for a prime p to be represented by the form $x^2 + 3y^2$.
13. Is there a positive definite binary quadratic form that represents 2 and the primes congruent to 1 or 3 modulo 8, but no other primes? Is there such a form representing the primes congruent to 1 modulo 4 only?
14. Find congruence conditions for a positive integer n to be properly represented by at least one of the two forms $x^2 + xy + 4y^2$ and $2x^2 + xy + 2y^2$.

    Assume that n is coprime to 15, and properly represented by at least one of the forms. Show that congruence conditions modulo 15 allow one to decide which form represents n.