Number Theory - Example Sheet 3.

1. Use Legendre's formula to compute $\pi(207)$.

2. Let N be an integer >1. Using the fact that the exact power of any prime p dividing N is $\sum_{k=1}^{\infty}\left[\frac{N}{p^k}\right]$, and the inequality $N! > \left(\frac{N}{e}\right)^N$ (which you should also justify), prove that

$$\sum_{p \leq N} \frac{\log p}{p-1} > \log N - 1.$$

3. Prove that every non-constant polynomial with integer coefficients assumes infinitely many composite values

4. Prove that every integer $N > 6$ can be expressed as a sum of distinct primes.

5. If p is a prime with $p \equiv 3 \bmod 8$, prove that $\sum_{r=1}^{\frac{p-1}{2}} \left(\frac{r}{p}\right)$ is always divisible by 3.

6. Prove that if there is a zero of $\zeta(s)$ on the line $R(s) = 1$, then it must be a simple zero.

7. Define $\Lambda(n)$ to be $\log p$ if n is a (positive) power of a prime p, and to be 0 otherwise. Prove that, for $R(s)>1$, we have

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

8. Prove that, for $R(s)>1$, we have

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s},$$

where $d(n)$ denotes the number of positive divisors of n, including 1 and n.

9. Prove that the forms
$$3x^2 + 2xy + 23y^2, \quad 2x^2 + 4xy + 5y^2$$
are not equivalent (under the action of $SL_2(\mathbb{Z})$).

10. Make a list of all reduced positive definite quadratic forms of discriminant $-d$, where
$$d = 8, 11, 12, 16, 19, 23, 163.$$

11. Find a necessary and sufficient condition for a positive integer n to be properly represented by at least one of the two forms
$$x^2 + xy + 4y^2, \quad 2x^2 + xy + 2y^2.$$

Assume n is prime to 15, and properly represented by at least one of the forms. Show that congruence conditions modulo 15 allow one to decide which form represents n.