Number Theory - Problem sheet 2

1. Evaluate the following Jacobi symbols (in fact, they are Legendre symbols):-

$$\left(\frac{20964}{1987}\right), \left(\frac{741}{9283}\right), \left(\frac{5}{160465489}\right), \left(\frac{3083}{3911}\right).$$

2. Find all odd primes p for which 15 is a quadratic residue mod p.

3. Prove that 3 is a quadratic non-residue modulo any Mersenne prime $2^n - 1$, with $n > 2$.

4. Let p be a prime with $p \equiv 1 \bmod 4$. Prove that the sum of the quadratic residues in the interval $[1, p-1]$ is equal to the sum of the quadratic non-residues in this interval.

5. Let a be a positive integer which is not a square. Prove that there are infinitely many odd primes p such that $\left(\frac{a}{p}\right) = -1$.

6. Find all ~~odd~~ bases for which 39 is an Euler pseudo-prime.

7. Let $N = (6t+1)(12t+1)(18t+1)$, where t is a positive integer such that all of $6t+1$, $18t+1$, $12t+1$ are prime numbers. Prove that N is a Carmichael number. Use this construction to find 3 Carmichael numbers.

8. Prove that there are 36 bases for which 91 is a pseudo-prime. In general, if both p and $2p-1$ are prime numbers, show that $N = p(2p-1)$ is a pseudo-prime for precisely half of all bases.

9. Let $N = 561$. Find the number of bases b for which N is an Euler pseudo-prime. Show there are precisely 10 bases for which N is a strong pseudo-prime.

10. Let p be a prime > 5. Prove that $N = (4^p+1)/5$ is always a composite integer. Prove that N is always a strong pseudo-prime to the base 2.

11. Assume that n is an integer >1 such that $F_n = 2^{2^n}+1$ is composite $(n = 5, \ldots)$. Prove that F_n is a pseudo-prime to the base 2.

12. Give an example of an odd composite integer N and a base b such that $b^{\frac{N-1}{2}} \equiv \pm 1 \bmod N$, but N is not an Euler pseudo-prime to the base b.