Number Theory – Problem Sheet 3

1. Use Legendre's formula to compute $\pi(207)$.

2. The polynomial $n^2 - n + 41$ is prime for all n with $0 \leqslant n \leqslant 40$. Check this for $n = 0, \ldots, 15$. Obviously $n = 41$ yields a composite value. Prove that the polynomial has infinitely many composite values.

3. For each integer $n \geqslant 1$ and each prime p, prove that the exact power of p dividing $n!$ is
$$\sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right].$$

4. If p is a prime with $p \equiv 3 \bmod 8$, prove that
$$\sum_{r=1}^{\frac{p-1}{2}} \left(\frac{r}{p}\right)$$
is always divisible by 3.

5. Prove that if there is a zero of $\zeta(s)$ on the line $R(s) = 1$, then it must be a simple zero.

6. Prove that, for $R(s) > 1$, we have
$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s},$$
where $\Lambda(n) = \log p$ if n is a positive power of a prime p, and $\Lambda(n) = 0$ otherwise.

7. Prove that, for $R(s) > 1$, we have

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s},$$

where $d(n)$ denotes the number of positive divisors of n, including 1 and n.

8. Make a list of all reduced positive definite quadratic forms of discriminant $-d$, where $d = 8, 11, 12, 16, 19, 23, 163$.

9. Find a necessary and sufficient condition for a~~n~~ positive integer n to be properly represented by at least one of the two forms

$$x^2 + xy + 4y^2, \quad 2x^2 + xy + 2y^2.$$

If we assume n is not divisible by 15, show that congruences mod 15 allow one to decide which form represents n, when n is properly represented by one of the two forms.