Number Theory – Problem sheet 3

1. Use Legendre's formula to compute $\pi(207)$.

2. Prove that, for $R(s) > 1$, we have
$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}$$
where $d(n)$ denotes the number of divisors of n, including 1 and n.

3. Prove that if there is a zero of $\zeta(s)$ on the line $R(s) = 1$, then it must be simple.

4. Prove that, for $R(s) > 1$, we have
$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s},$$
where $\Lambda(n) = \log p$ if n is a power of a prime p, and $\Lambda(n) = 0$ otherwise.

5. Make a list of all reduced positive definite quadratic forms of discriminant $-d$, where $d = 8, 11, 12, 15, 16, 19, 20, 23$.

6. Determine necessary and sufficient conditions for an odd positive integer n, prime to 15, to be properly represented by $x^2 + xy + 11y^2$

7. If p is a prime with $p \equiv 3 \bmod 8$, prove that $\sum_{r=1}^{\frac{p-1}{2}} \left(\frac{r}{p}\right)$ is always divisible by 3.

8. Let $f_1(x,y) = a_1x^2 + b_1xy + c_1y^2$ and $f_2(x,y) = a_2x^2 + b_2xy + c_2y^2$ be two reduced positive definite quadratic forms which are equivalent, say

$$f_2(x,y) = f_1(rx+sy, tx+uy) \text{ with } ru - ts = 1.$$

The following sequence of escercises will lead you through the proof that $f_1 \equiv f_2$. We may assume $a_1 \geqslant a_2$. Prove that

(i) $a_2 \geqslant a_1r^2 - a_1|rt| + a_1t^2$;

(ii) $a_1 \geqslant a_2 \geqslant a_1|rt|$;

(iii) $a_1 = a_2$;

(iv) If $a_1 = c_1$ and $a_2 = c_2$, then $b_1 = b_2$;

(v). Assuming $c_1 > a_1$, we have $t = 0$;

(vi). Deduce that $b_2 \equiv b_1 \bmod 2a$, and conclude that $b_2 = b_1$ and $c_2 = c_1$.