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Number Fields Examples

Michaelmas Term 2002

Sheet 2

1. Show that, in the field $\mathbb{Q}(\sqrt{-6})$, the ideal $[2]$ factorises as $[2, \sqrt{-6}]^2$ as a product of prime ideals. Factorise the ideal $[6]$ similarly.

2. Find single generators for the ideals $[2613, 2171]$ in $\mathbb{Z}$ and $[51 - 5i, 43 + 7i]$ in the Gaussian field $\mathbb{Q}(i)$.

3. Find the fundamental unit in $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ and hence describe all the integer solutions of the equations
$$x^2 - 2y^2 = \pm 1, \quad x^2 - 3y^2 = \pm 1.$$

4. Find the fundamental unit in $\mathbb{Q}(\sqrt{5})$. Verify that $[11] = [4 + \sqrt{5}][4 - \sqrt{5}]$ as a product of prime ideals. Hence describe all the integer solutions of the equations
$$x^2 - 5y^2 = 11, \quad x^2 - 5y^2 = 121.$$
Are there any integer solutions x, y of the equation $x^2 - 5y^2 = 7$?

5. Find the fundamental unit in $\mathbb{Q}(\sqrt{7})$. Describe all the integer solutions of the equations
$$x^2 - 7y^2 = 2, \quad x^2 - 7y^2 = 13.$$

6. Show that if a number field K has degree $d = s + 2t$ where s is the number of real conjugate fields and $2t$ is the number of complex conjugate fields, then the discriminant D of K satisfies $(-1)^t D > 0$.

7. (Tripos 94) Find a fundamental unit in the ring of integers of $\mathbb{Q}(\sqrt{10})$.
Describe all integer solutions of the equation $x^2 - 10y^2 = n$ for $n = -1, 6$ and 7.

8. (Tripos 96) Give an account of the theory of units in algebraic number fields with particular reference to the quadratic field.
By considering the expression $(1-\zeta^j)/(1-\zeta)$ and noting that $\zeta^{1/2} = \pm\zeta^{(q+1)/2}$ show that the elements
$$\sin(j\pi/q)/\sin(\pi/q), \quad j = 2, \cdots, \tfrac{1}{2}(q-1),$$
are real, positive units in the cyclotomic field $\mathbb{Q}(\zeta)$, where $\zeta = e^{2\pi i/q}$ and q is an odd prime.

9. (Tripos 97–adapted) Show that each prime ideal in an algebraic number field k divides a unique rational prime p. Establish the equation

$$e_1 f_1 + \cdots + e_r f_r = n$$

relating the ramification indices of the distinct prime ideal factors of p and their degrees, where n is the degree of k. Deduce that p can have at most n such factors and specify when this bound is attained in the case of the quadratic field $\mathbb{Q}(\sqrt{d})$ with $d \equiv 2$ or $3 \pmod 4$.

10. (Tripos 97–adapted) Give a simple reason as to why the equation $x^2 - 11y^2 = -1$ has no solution in integers x, y. Hence verify that $\epsilon = 10 + 3\sqrt{11}$ is the smallest unit in $k = \mathbb{Q}(\sqrt{11})$ exceeding 1. What does this imply about the group of units in k?

    Write down an expression for $x - \sqrt{11}y$ that gives the complete solution in integers x, y of the equation $x^2 - 11y^2 = -2$.

11. Using the fact that the Gaussian field has unique factorisation, find all solutions in integers x, y of the equation $y^2 + 1 = x^3$. Similarly, using the fact that the field $\mathbb{Q}(\sqrt{-2})$ has unique factorisation, find all solutions in integers x, y of the diophantine equation $y^2 + 2 = x^3$.

12. Show that the ring $\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$ is Euclidean whence the field $\mathbb{Q}(\sqrt{-7})$ has unique factorisation.

    Find all solutions in integers x, y of the diophantine equation $y^2 + y + 2 = x^3$.