<u>Notation</u>. If K is a finite extension of $\mathbb{Q}$, d_K will denote the degree of K over $\mathbb{Q}$, and $\mathcal{O}_K$ the ring of integers of K.

1. Let $K = \mathbb{Q}(\theta)$, where θ is any root of $X^3 - 5X + 3$. Determine d_K, $\mathcal{O}_K$, and the discriminant of K. Determine the factorization into prime ideals of $p\,\mathcal{O}_K$, where $p = 2, 3, 5, 7, 11$. Determine all non-zero ideals $\mathfrak{a}$ of $\mathcal{O}_K$ with $N_{K/\mathbb{Q}}\mathfrak{a} \leq 11$.

2. Let $K = \mathbb{Q}(\sqrt{11})$. Prove that $11\,\mathcal{O}_K = \wp^2$, where $\wp$ is a prime ideal of $\mathcal{O}_K$. If p is a prime number with $p \neq 11$, prove that $p\,\mathcal{O}_K = \wp_1 \wp_2$, where $\wp_1$ and $\wp_2$ are prime ideals of $\mathcal{O}_K$ if and only if 11 is a quadratic residue modulo p.

3. Prove that $f(X) = X^5 - X + 1$ is irreducible over $\mathbb{Q}$ (hint: consider its reduction modulo 5). Let $K = \mathbb{Q}(\theta)$, where θ is a root of $f(X)$. Determine $\mathcal{O}_K$, and the discriminant of K. Determine the prime factorization of $2\,\mathcal{O}_K$, $3\,\mathcal{O}_K$, $5\,\mathcal{O}_K$.

4. Prove that the polynomial $X^3 + X^2 - 2X + 8$ is irreducible over $\mathbb{Q}$, and let $K = \mathbb{Q}(\theta)$, where θ is one of its roots. Prove that (i). $D(1, \theta, \theta^2) = 4.503$, (ii). $\varphi = 4/\theta$ belongs to $\mathcal{O}_K$, but does not belong to $\mathbb{Z}[\theta]$, (iii). $A = \mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\varphi$ is a subring of K which contains $\mathbb{Z}[\theta]$ properly, and (iv) $A = \mathcal{O}_K$ and the discriminant of K is 503.

5. Give an example of an integral domain R containing a non-zero prime ideal $\wp$, and a subring R' of R such that $\wp \cap R' = (0)$

6. Give an example of a number field K such that $\mathcal{O}_K$ is not a principal ideal domain, justifying carefully your answer.

7. Let K be a number field, and $\wp$ a non-zero prime ideal of $\mathcal{O}_K$. Prove that, for all integers $n \geq 1$, $\mathcal{O}_K/\wp^n$ is a ring in which every ideal is principal.
(Hint: show that the only proper ideals of $\mathcal{O}_K/\wp^n$ are $\wp/\wp^n, \ldots, \wp^{n-1}/\wp^n$. Choose π in $\wp \setminus \wp^2$, and show that $\wp^\nu = \mathcal{O}_K \pi^\nu + \wp^n$ when $\nu = 1, \ldots, n-1$).