## Number Fields IID, Lent 2021: Example Sheet 1 of 3

1. Find the minimal polynomials over $\mathbb{Q}$ of

$$(1+i)\sqrt{3}, \qquad i+\sqrt{3}, \qquad 2\cos(2\pi/7).$$

2. Which of the following are algebraic integers?

$$\sqrt{5}/\sqrt{2},\ (1+\sqrt{3})/2,\ (\sqrt{3}+\sqrt{7})/2,\ \frac{3+2\sqrt{6}}{1-\sqrt{6}},\ (1+\sqrt[3]{10}+\sqrt[3]{100})/3,\ 2\cos(2\pi/19).$$

3. Let $d>1$ be an integer. Show that the only units in the ring

$$\mathbb{Z}[\sqrt{-d}] = \{a+b\sqrt{-d} : a,b\in\mathbb{Z}\}$$

are $\pm 1$.

4. Let $K$ be a number field. Show that every extension $L/K$ of degree 2 is of the form $L=K(\sqrt{a})$ with $a\in K^*$, $a\notin (K^*)^2$. Show further that there is a isomorphism $K(\sqrt{a})\cong K(\sqrt{b})$ inducing the identity on $K$ if and only if $a/b\in (K^*)^2$.

5. Let $K=\mathbb{Q}(\theta)$ where $\theta$ is a root of $X^3-2X+6$. Show that $[K:\mathbb{Q}]=3$ and compute $N_{K/\mathbb{Q}}(\alpha)$ and $\mathrm{tr}_{K/\mathbb{Q}}(\alpha)$ for $\alpha=n-\theta$, $n\in\mathbb{Z}$ and $\alpha=1-\theta^2, 1-\theta^3$.

6. Let $K=\mathbb{Q}(\delta)$ where $\delta=\sqrt[3]{d}$ and $d\neq 0,\pm1$ is a square-free integer. Show that $\mathrm{disc}(1,\delta,\delta^2)=-27d^2$. By calculating the traces of $\theta$, $\delta\theta$, $\delta^2\theta$, and the norm of $\theta$, where $\theta=u+v\delta+w\delta^2$ with $u,v,w\in\mathbb{Q}$, show that the ring of integers $\mathcal{O}_K$ of $K$ satisfies

$$\mathbb{Z}[\delta]\subset\mathcal{O}_K\subset\tfrac{1}{3}\mathbb{Z}[\delta].$$

7. Let $f(X)\in\mathbb{Q}[X]$ be a monic irreducible polynomial of degree $n$, and $\theta\in\mathbb{C}$ a root of $f$.

   (i) Show that $\mathrm{disc}(f)=(-1)^{\binom{n}{2}}N_{K/\mathbb{Q}}(f'(\theta))$ where $K=\mathbb{Q}(\theta)$.

   (ii) Let $f(X)=X^n+aX+b$. Write down the matrix representing multiplication by $f'(\theta)$ with respect to the basis $1,\theta,\dots,\theta^{n-1}$ for $K$. Hence show that
   $$\mathrm{disc}(f)=(-1)^{\binom{n}{2}}((1-n)^{n-1}a^n+n^nb^{n-1}).$$

8. Compute an integral basis for $\mathcal{O}_K$ in the cases $K=\mathbb{Q}[X]/(X^3+X+1)$ and $K=\mathbb{Q}[X]/(X^3-X-4)$.

9. Suppose that $K=\mathbb{Q}(\theta)$ is a number field of degree $n=r+2s$ in the usual notation ($r$ is the number of real embeddings of $K$ and $s$ the number of pairs of complex conjugate embeddings). Show that the sign of the discriminant of $K$ is $(-1)^s$.

 a.j.scholl@dpmms.cam.ac.uk

10. Let $K = \mathbb{Q}(i, \sqrt{2})$. By computing the relative traces $\operatorname{tr}_{K/k}(\theta)$ where $k$ runs through the three quadratic subfields of $K$, show that the algebraic integers $\theta$ in $K$ have the form $\frac{1}{2}(\alpha + \beta\sqrt{2})$, where $\alpha = a + ib$ and $\beta = c + id$ are Gaussian integers. By considering $N_{K/k}(\theta)$ where $k = \mathbb{Q}(i)$ show that

$$\begin{aligned} a^2 - b^2 - 2c^2 + 2d^2 &\equiv 0 \pmod 4, \\ ab - 2cd &\equiv 0 \pmod 2. \end{aligned}$$

Hence prove that an integral basis for $\mathcal{O}_K$ is $1, i, \sqrt{2}, \frac{1}{2}(1+i)\sqrt{2}$, and calculate the discriminant of $K$.

11. Let $K$ be a quadratic field and $I \subset \mathcal{O}_K$ an ideal. Show that $I = (\alpha, \beta)$ for some $\alpha \in \mathbb{Z}$ and $\beta \in \mathcal{O}_K$. Let $c = \gcd(\alpha^2, \alpha \operatorname{tr}_{K/\mathbb{Q}}\beta, N_{K/\mathbb{Q}}\beta)$. By computing the norm and trace show that $\frac{\alpha\beta}{c} \in \mathcal{O}_K$. Deduce that $(\alpha, \beta)(\alpha, \beta')$ is principal, where $\beta\beta' = N_{K/\mathbb{Q}}\beta$.

 a.j.scholl@dpmms.cam.ac.uk