

Number Fields: Example Sheet 1 of 3

1. Find the minimal polynomials over \mathbb{Q} of

$$(1+i)\sqrt{3}, \quad i+\sqrt{3}, \quad 2\cos(2\pi/7).$$

2. Which of the following are algebraic integers?

$$\sqrt{5}/\sqrt{2}, \quad (1+\sqrt{3})/2, \quad (\sqrt{3}+\sqrt{7})/2, \quad \frac{3+2\sqrt{6}}{1-\sqrt{6}}, \quad (1+\sqrt[3]{10}+\sqrt[3]{100})/3, \quad 2\cos(2\pi/19).$$

3. Let $d > 1$ be an integer. Show that the only units in the ring

$$\mathbb{Z}[\sqrt{-d}] = \{a + b\sqrt{-d} : a, b \in \mathbb{Z}\}$$

are ± 1 .

4. (i) Explain why the equations

$$2 \cdot 11 = (5 + \sqrt{3})(5 - \sqrt{3})$$

and

$$(2 + \sqrt{7})(3 - 2\sqrt{7}) = (5 - 2\sqrt{7})(18 + 7\sqrt{7})$$

are not inconsistent with the fact $\mathbb{Z}[\sqrt{3}]$ and $\mathbb{Z}[\sqrt{7}]$ have unique factorisation.

(ii) Find equations to show that $\mathbb{Z}[\sqrt{d}]$ is not a UFD for $d = -10, -13, -14$.

5. Let K be a field with $\text{char}(K) \neq 2$. Show that every extension L/K of degree 2 is of the form $L = K(\sqrt{a})$ with $a \in K^*$, $a \notin (K^*)^2$. Show further that $K(\sqrt{a}) = K(\sqrt{b})$ if and only if $a/b \in (K^*)^2$.

6. Let $A \subseteq B \subseteq C$ be rings.

(i) Show that if B is finite over A , and C is finite over B , then C is finite over A .
(ii) Show that if B is integral over A , and C is integral over B , then C is integral over A .

Now let $\mathbb{Q} \subseteq K \subseteq L$ be finite extensions of fields.

(i) Show that if $\alpha \in L$ is integral over \mathcal{O}_K it is an algebraic integer.
(ii) Show that if $f \in K[x]$ is monic, and $f^n \in \mathcal{O}_K[x]$ for some n , then $f \in \mathcal{O}_K[x]$.

7. Let $K = \mathbb{Q}(\theta)$ where θ is a root of $X^3 - 2X + 6$. Show that $[K : \mathbb{Q}] = 3$ and compute $N_{K/\mathbb{Q}}(\alpha)$ and $\text{Tr}_{K/\mathbb{Q}}(\alpha)$ for $\alpha = n - \theta$, $n \in \mathbb{Z}$ and $\alpha = 1 - \theta^2, 1 - \theta^3$.

8. Let $K = \mathbb{Q}(\delta)$ where $\delta = \sqrt[3]{d}$ and $d \neq 0, \pm 1$ is a square-free integer. Show that $\Delta(1, \delta, \delta^2) = -27d^2$. By calculating the traces of $\theta, \delta\theta, \delta^2\theta$, and the norm of θ , where $\theta = u + v\delta + w\delta^2$ with $u, v, w \in \mathbb{Q}$, show that the ring of integers \mathcal{O}_K of K satisfies

$$\mathbb{Z}[\delta] \subset \mathcal{O}_K \subset \frac{1}{3}\mathbb{Z}[\delta].$$

9. Let $K = \mathbb{Q}(\alpha)$ be a number field. Suppose $\alpha \in \mathcal{O}_K$ and let $f \in \mathbb{Z}[X]$ be its minimal polynomial.

- (i) Show that if the discriminant of f is a square-free integer then $\mathcal{O}_K = \mathbb{Z}[\alpha]$.
- (ii) Compute an integral basis for K in the cases $f(X) = X^3 + X + 1$ and $f(X) = X^3 - X - 4$.

[The discriminant of $X^3 + aX + b$ is $-4a^3 - 27b^2$.]

10. Let $K = \mathbb{Q}(i, \sqrt{2})$. By computing the relative traces $\text{Tr}_{K/k}(\theta)$ where k runs through the three quadratic subfields of K , show that the algebraic integers θ in K have the form $\frac{1}{2}(\alpha + \beta\sqrt{2})$, where $\alpha = a + ib$ and $\beta = c + id$ are Gaussian integers. By considering $N_{K/k}(\theta)$ where $k = \mathbb{Q}(i)$ show that

$$\begin{aligned} a^2 - b^2 - 2c^2 + 2d^2 &\equiv 0 \pmod{4}, \\ ab - 2cd &\equiv 0 \pmod{2}. \end{aligned}$$

Hence prove that an integral basis for K is $1, i, \sqrt{2}, \frac{1}{2}(1+i)\sqrt{2}$, and calculate the discriminant D_K .

11. Suppose that K is a number field of degree $n = r+2s$ in the usual notation (r is the number of real embeddings of K and s the number of pairs of complex conjugate embeddings). Show that the sign of the discriminant D_K is $(-1)^s$.

12. Let $f(X) \in \mathbb{Q}[X]$ be an irreducible polynomial of degree n , and $\theta \in \mathbb{C}$ a root of f .

- (i) Show that $\text{disc}(f) = (-1)^{\binom{n}{2}} N_{K/\mathbb{Q}}(f'(\theta))$ where $K = \mathbb{Q}(\theta)$.
- (ii) Let $f(X) = X^n + aX + b$. Write down the matrix representing multiplication by $f'(\theta)$ with respect to the basis $1, \theta, \dots, \theta^{n-1}$ for K . Hence show that

$$\text{disc}(f) = (-1)^{\binom{n}{2}} ((1-n)^{n-1} a^n + n^n b^{n-1}).$$

The following extra questions are just for fun. They can be answered using material from the Part IB course *Groups Rings and Modules*.

12. Let $\omega \neq 1$ be a cube root of unity, and let $p \neq 3$ be a prime.

- (i) By considering units in $\mathbb{Z}[\omega]$ show that $x^2 + 3y^2$ represents p if and only if $x^2 + xy + y^2$ represents p .
- (ii) Use that \mathbb{F}_p^* is cyclic to find a condition on p for the congruence $x^2 + x + 1 \equiv 0 \pmod{p}$ to be soluble.
- (iii) Use unique factorisation in $\mathbb{Z}[\omega]$ to determine the set of primes in (i).

13. Show that the rings $\mathbb{Z}[i]$ and $\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$ are Euclidean. Hence find all integer solutions to the equations $y^2 = x^3 - 4$ and $y^2 + y = x^3 - 2$.

14. Let $n \geq 3$ be an integer. Suppose $f, g, h \in \mathbb{C}[X]$ are coprime polynomials satisfying $f^n + g^n = h^n$. Use unique factorisation in $\mathbb{C}[X]$ to construct a new solution to this equation involving polynomials of smaller degree. Deduce that f, g, h must be constant.