Logic & Set Theory
Lent Term 2026
Part II of the Mathematical Tripos
University of Cambridge
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Example Sheet #1

(1) Show that there is no Boolean algebra with three elements, but there is one with four elements. What about Boolean algebras with five, six, seven, or eight elements?

(2) Let $\mathbf{B} = (B, +, \cdot, \to, -, 0, 1)$ be a Boolean algebra. In $\mathbf{B}$, show that $x \cdot y = x$ if and only if $x + y = y$. Define $x \leq y :\iff x + y = y$ and show that $\leq$ is a partial order (i.e., reflexive, transitive, and antisymmetric). Show that if $x \leq y$, then $x \to y = 1$.

(3) Let $\mathbf{3}$ be any reasoning algebra with three elements $\{0, 1, 2\}$ such that $\mathbf{2}$ is a substructure of $\mathbf{3}$. Consider *excluded middle* $(x + -x = 1)$, *idempotence for* $+$ $(x + x = x)$, *idempotence for* $\cdot$ $(x \cdot x = x)$, *double negation* $(--x = x)$, and *contradiction* $(x \cdot -x = 0)$.

Show in $\mathbf{3}$ that double negation implies that either idempotence for $+$ or excluded middle fails; show that double negation implies that either idempotence for $\cdot$ or contradiction fails.

(4) Let Σ be a nonempty set, F be a set of functions $f\colon (\Sigma^*)^* \to \Sigma^*$, and $A \subseteq \Sigma^*$. We say that $X \subseteq \Sigma^*$ is *F-closed* if for all $f \in F$ and $s \in X^*$, we have that $f(s) \in X$. Define recursively

$$A_0 := A,$$
$$A_{i+1} := A_i \cup \{f(s)\,;\, s \in A_i^*,\ f \in F\},$$
$$A_\infty := \bigcup_{i \in \mathbb{N}} A_i$$

and show that
$$A_\infty = \bigcap \{X\,;\, X \supseteq A \text{ is } F\text{-closed}\}.$$

Discuss how this relates to the fact that we can prove statements about Fml by induction.

(5) For $p_1, p_2, p_3 \in P$, which of the following formulas are tautologies?

(i) $((p_1 \Rightarrow (p_2 \Rightarrow p_3)) \Rightarrow (p_2 \Rightarrow (p_1 \Rightarrow p_3)))$;
(ii) $(((p_1 \vee p_2) \wedge (p_1 \vee p_3)) \Rightarrow (p_2 \vee p_3))$;
(iii) $((p_1 \Rightarrow \neg p_2) \Rightarrow (p_2 \Rightarrow \neg p_1))$.

(6) We call a formula a *literal* if it is either p or $\neg p$ for a propositional letter $p \in P$. We say that a formula is in *disjunctive normal form* if it is a disjunction of conjunctions of literals.

Show that every formula is equivalent to one in disjunctive normal form. Describe an algorithm that determines the equivalent formula in disjunctive normal form.

(7) Assume that $\varphi \models \psi$. Find $\iota \in \text{Fml}$ with $\varphi \models \iota$ and $\iota \models \psi$ such that all propositional letters occurring in ι occur in both φ and ψ.

(8) Two sets S and T of formulas are *equivalent* if $S \models T$ and $T \models S$. A set S of formulas is *independent* if for every $\varphi \in S$ we have $S \backslash \{\varphi\} \not\models \varphi$.

Show that every finite set of formulas has an independent subset equivalent to it. Give an infinite set of formulas that has no independent subset equivalent to it. Show that for every set of formulas there exists an independent set equivalent to it.

(9) There are $2^4 = 16$ binary operations on $2 = \{0, 1\}$. Let B be the set of these operations. A subset $X \subseteq B$ is *functionally complete* if all operations in B are equationally definable from operations in X.

Show that $L = \{+, \cdot, \rightarrow, -, 0, 1\}$ is functionally complete. Explain why results from the lectures imply that $\{\rightarrow, 0\}$ is also functionally complete. Are there any one-element subsets of L that are functionally complete? Is there an operation $\star \in B$ such that $\{\star\}$ is functionally complete?

(10) Assume that there is a non-empty finite set I of people and each person $i \in I$ believes that the formulas in $B_i \subseteq \mathrm{Fml}$ are true. Assume that the set B_i is consistent for each $i \in I$. Show that the set of propositions that they all believe to be true is also consistent. Must the set of propositions that a strict majority believe be consistent?

(11) Let $T \subseteq \mathrm{Fml}$. We call a T-proof *implicative* if it does not use any axioms of type 4 and we call T *implicatively consistent* if there is no implicative T-proof of $\bot$. Show that $\{p, (p \Rightarrow \bot)\}$ is implicatively inconsistent and $\{p, \neg p\}$ is implicatively consistent.

(12) We say that a triple (μ_0, μ_1, μ_2) is an *instance of conjunction introduction* if there are formulas φ and ψ such that $\mu_0 = \varphi$, $\mu_1 = \psi$, and $\mu_2 = (\varphi \wedge \psi)$. Modify the definition of "*$\varphi_0 \ldots \varphi_n$ is a T-proof*" in the following way: for each $i \leq n$, the formula φ_i either satisfies condition (i), (ii), or (iii) from the definition in the lectures or

(iv) there are $j, k < i$ such that $(\varphi_j, \varphi_k, \varphi_i)$ is an instance of conjunction introduction.

Let us call this a *modified T-proof.*

Show that $T \vdash \varphi$ if and only if there is a modified T-proof of φ. Can you prove this without using the completeness theorem? Can you think of other modifications of the notion of proof with the same property?

(13) *Kőnig's Lemma* says that every countably infinite and finitely branching tree has an infinite branch. Prove Kőnig's Lemma by using the compactness theorem for propositional logic.