Lent 2022 **LOGIC AND SET THEORY – EXAMPLES 2** IBL

1. Write down subsets of the reals that have order-types $\omega + \omega$, ω^2 and ω^3.

2. Let α and β be non-zero ordinals. Must we have $\alpha + \beta > \alpha$? Must we have $\alpha + \beta > \beta$?

3. Is there a non-zero ordinal α with $\alpha\omega = \alpha$? What about $\omega\alpha = \alpha$?

4. Show that the inductive and the synthetic definitions of ordinal multiplication coincide.

5. Let α, β, γ be ordinals. Prove that $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

6. Let α, β, γ be ordinals. Must we have $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$? Must we have $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$?

7. Let α and β be ordinals with $\alpha \geq \beta$. Show that there is a unique ordinal γ such that $\beta + \gamma = \alpha$. Must there exist an ordinal γ with $\gamma + \beta = \alpha$?

8. Find two totally ordered sets such that neither is isomorphic to a subset of the other. Can you find three such sets?

9. Let α be a countable (non-zero) limit ordinal. Prove that there exists an increasing sequence $\alpha_1 < \alpha_2 < \alpha_3 < \ldots$ with supremum equal to α. Is this result true for $\alpha = \omega_1$?

10. Show that, for every countable ordinal α, there is a subset of $\mathbb{Q}$ of order-type α. Why is there no subset of $\mathbb{R}$ of order-type ω_1?

11. An ordinal written as $\omega^{\alpha_1} n_1 + \ldots + \omega^{\alpha_k} n_k$, where $\alpha_1 > \ldots > \alpha_k$ are ordinals (and k and $n_1, \ldots, n_k$ are non-zero natural numbers), is said to be in *Cantor Normal Form.* Show that every non-zero ordinal has a unique Cantor Normal Form. What is the Cantor Normal Form for the ordinal ϵ_0?

12. What is the smallest fixed point of $\alpha \mapsto \omega^\alpha$? The next smallest? And the next smallest? Show that the fixed points are unbounded, and explain why this means that we may index the fixed points by the ordinals. Is there a countable ordinal α such that α is the α-th fixed point?

13. A train stops at station α, each $0 \leq \alpha \leq \omega_1$. It leaves station 0 empty. When it arrives at station α ($0 < \alpha < \omega_1$), one person gets off (unless it is empty), and then some countable number of people get on. What are the possible numbers of people on the train as it arrives at station ω_1? (Each person who makes a journey gets on at some station and gets off at some later station.)

$^+$14. Let X be a totally ordered set such that the only order-preserving injection from X to itself is the identity. Must X be finite?