Lent 2020 **LOGIC AND SET THEORY – EXAMPLES 1** PAR

1. Which of the following propositions are tautologies?
(i) $(p_1 \Rightarrow (p_2 \Rightarrow p_3)) \Rightarrow (p_2 \Rightarrow (p_1 \Rightarrow p_3))$
(ii) $((p_1 \vee p_2) \wedge (p_1 \vee p_3)) \Rightarrow (p_2 \vee p_3)$
(iii) $(p_1 \Rightarrow (\neg p_2)) \Rightarrow (p_2 \Rightarrow (\neg p_1))$

2. Use the Deduction Theorem to show that $p \vdash \neg\neg p$.

3. Show that $\{p, q\} \vdash p \wedge q$ in three different ways: by writing down a proof, by using the Deduction Theorem, and by using the Completeness Theorem.

4. Give propositions p and q for which $(p \Rightarrow q) \Rightarrow \neg(q \Rightarrow p)$ is a tautology.

5. An alternative version of the Compactness Theorem says that if $S \models p$ then there is some finite subset $T \subset S$ with $T \models p$. Give two proofs of this: (i) by deducing it from the Compactness Theorem; and (ii) directly from the Completeness Theorem.

6. Three people each have a set of beliefs: a consistent deductively closed set. Show that the set of propositions that they all believe is also consistent and deductively closed. Must the set of propositions that a majority believe be consistent? Must it be deductively closed?

7. Show that the third axiom cannot be deduced from the first two. In other words, show that (for some p) there is no proof of $(\neg\neg p) \Rightarrow p$ that uses only the first two axioms and modus ponens.

8. Let $t_1, t_2, \ldots$ be propositions such that, for every valuation v, there exists n with $v(t_n) = 1$. Show that there is some n for which $\vdash t_1 \vee t_2 \vee \cdots \vee t_n$.

9. Taking on trust for now that the informal method used to prove the Completeness Theorem when the set of primitive propositions is allowed to be uncountable both makes sense and works, prove similarly that every vector space over $\mathbb{R}$ has a basis.

10. Let a and c be propositions such that $a \vdash c$. Show that there is a proposition b, in which the only primitive propositions appearing are those that appear in both a and c, such that $a \vdash b$ and $b \vdash c$.

11. Two sets S, T of propositions are *equivalent* if $S \vdash t$ for every $t \in T$ and $T \vdash s$ for every $s \in S$. A set S of propositions is *independent* if for every $s \in S$ we have $S - \{s\} \nvdash s$. Show that every finite set of propositions has an independent subset equivalent to it. Give an infinite set of propositions that has no independent subset equivalent to it. Show, however, that for every countable set of propositions there exists an independent set equivalent to it.

12. Let p be a tautology not involving the symbol $\perp$. Must it be possible to prove p without using the third axiom?