1. Show that $\{\mathbf{Inf}, \mathbf{Sep}\} \vdash \mathbf{Emp}$. Does $\mathbf{Inf} \vdash \mathbf{Emp}$? Does $\mathbf{Sep} \vdash \mathbf{Emp}$? *[N.B. You should interpret* $\{\mathbf{Inf}, \mathbf{Sep}\}$ *as "the collection of formulae in the language of sets consisting of the Axiom of Infinity and every instance of the Axiom of Separation" etc.]*

2. Show that $\mathbf{Rep} \vdash \mathbf{Sep}$. Show also that $\{\mathbf{Emp}, \mathbf{Pow}, \mathbf{Rep}\} \vdash \mathbf{Pair}$.

3. Write down sentences in the language of sets to express the assertions that, for any two sets $x$ and $y$, the product $x \times y$ and the set of all functions from $x$ to $y$ exist. Indicate how to deduce these sentences from the axioms of ZF.

4. Write down a formula $p$ in the language of sets with $\mathrm{FV}(p) = \{x\}$ that says "$x$ is a (von Neumann) ordinal". What should the von Neumann ordinal $\omega^2$ be? Why do the axioms of ZF prove that it is a set?

5. Is it true that if $x$ is a transitive set then the relation $\in$ on $x$ is a transitive relation? Does the converse hold?

6. Let $\in$-**Ind** denote the principle of $\in$-induction and let $p$ be the formula $(\forall y)((x \in y) \Rightarrow (\exists z)((z \in y) \wedge (z \cap y = \emptyset)))$. Show that $(\mathrm{ZF}\setminus\{\mathbf{Fdn}\}) \cup \{\in\!-\mathbf{Ind}\} \vdash (\forall x)p$ and hence that $(\mathrm{ZF}\setminus\{\mathbf{Fdn}\}) \cup \{\in\!-\mathbf{Ind}\} \vdash \mathbf{Fdn}$.

7. What is the rank of $\{2, 3, 6\}$? What is the rank of $\{\{2, 3\}, \{6\}\}$? Work out the ranks of $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, using your favourite constructions of these objects from $\omega$.

8. A set $x$ is called *hereditarily finite* if each member of $\mathrm{TC}(\{x\})$ is finite. Prove that the class HF of hereditarily finite sets coincides with $V_\omega$. Which of the axioms of ZF are satisfied in the structure HF?

9. Which of the axioms of ZF are satisfied in the structure $V_{\omega+\omega}$?

10. If ZF is consistent then, by Downward Löwenheim-Skolem, it has a countable model. Doesn't this contradict the fact that, for example, the power-set of $\omega$ is uncountable?

11. Assume that ZF is consistent. We extend the language of ZF by adding new constants $\alpha_1, \alpha_2, \ldots$, and extend the axioms of ZF by adding (for each $n$) the assertions that $\alpha_n$ is an ordinal and that $\alpha_{n+1} < \alpha_n$. Explain why this theory has a model. In this model of ZF, haven't we contradicted the fact that the ordinals are "well-ordered" (that is to say, that each non-empty set of ordinals has a least element)?

12. Prove (in ZF) that a countable union of countable sets cannot have cardinality $\aleph_2$.

13. Is every countable model of PA isomorphic to $\mathbb{N}$? (What does "isomorphic" mean?)

14. Is $\mathrm{PA} \cup \{\neg\mathrm{Con}(\mathrm{PA})\}$ consistent? Is it $\omega$-consistent?

$^+$15. Show that the function $f(n) = 2^n$ is definable in the language of PA; that is to say, show that there is a formula $p$ in the language of PA with $FV(p) = \{x, y\}$ such that, for all $m$, $n \in \mathbb{N}$, $(m, n) \in p_{\mathbb{N}}$ iff $n = 2^m$.