Lent 2020 **LOGIC AND SET THEORY – EXAMPLES 3** PAR

1. Use Zorn's Lemma to show that every partial order may be extended to a total order.

2. Give a direct proof of Zorn's Lemma (not using ordinals and not using the Axiom of Choice) for countable posets.

3. Use Zorn's Lemma to prove one or more of the following statements (if at least one of them makes sense to you):
(i) Every commutative ring with a 1 has a maximal ideal;
(ii) Every Hilbert space has an orthonormal basis;
(iii) Let G be an infinite bipartite graph with parts X and Y such that every $x \in X$ has finite degree and, for all $A \subset X$, $|\Gamma(A)| \geqslant |A|$. Show that G contains a matching from X to Y. *(Don't attempt this one unless you know Hall's Theorem.)*

4. Was the Axiom of Choice used at any point in Chapter 2 (Ordinals)? If so, where?

5. Prove the Bourbaki-Witt theorem without using the Axiom of Choice.

6. Show that we must use the Axiom of Choice in any proof that, given any two sets, there is an injection from one of them into the other.

7. Let κ, λ and μ be cardinals. Show that $\kappa^\lambda \kappa^\mu = \kappa^{\lambda+\mu}$.

8. What is the cardinality of the set of all functions from $\mathbb{R}$ to $\mathbb{R}$? What is the cardinality of the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$?

9. Explain why, for each $n \in \mathbb{N}$, there is no surjection from $\aleph_n$ to $\aleph_{n+1}$. Use this fact to show that there is no surjection from $\aleph_\omega$ to ${\aleph_\omega}^{\aleph_0}$, and deduce that $2^{\aleph_0} \neq \aleph_\omega$.

10. Formulate sets of axioms in suitable languages (to be specified) for the following theories.
(i) The theory of fields of characteristic 2
(ii) The theory of posets having no maximal element
(iii) The theory of bipartite graphs
(iv) The theory of algebraically closed fields
(v) The theory of groups of order 60
(vi) The theory of simple groups of order 60
(vii) The theory of real vector spaces

11. Write down axioms (in the language of posets) for the theory of total orders that are dense (between any two elements is a third) and have no greatest or least element. Show that every countable model of this theory is isomorphic to $\mathbb{Q}$. Why does it follow that this theory is complete?

12. Show that the theory of fields of positive characteristic is not axiomatizable (in the language of fields), and that the theory of fields of characteristic zero is axiomatizable but not finitely axiomatizable.

13. Write down axioms, in a suitable language, for the theory of groups that have an element of infinite order. Can this be done in the language of groups?

14. Let L be the language consisting of a single function symbol f, of arity 1. Write down a theory T that asserts that f is a bijection with no finite orbits, and describe the countable models of T. Prove that T is a complete theory.

$^+$15. Let $\mathbb{A}$ denote the subfield of $\mathbb{C}$ consisting of the algebraic numbers. If a sentence (in the language of fields) holds in $\mathbb{C}$, must it also hold in $\mathbb{A}$?