Lent 2020 **LOGIC AND SET THEORY – EXAMPLES 2** PAR

1. Ordered pairs should satisfy $(x,y) = (z,u)$ iff $x = z$ and $y = u$. Show that, if we want, we can think of ordered pairs as "living in the universe of sets" by defining $(x,y) = \{\{x\},\{x,y\}\}$.

2. Write down subsets of the reals that have order-types $\omega+\omega+\omega$, ω^2 and ω^3.

3. Let α, β and γ be ordinals. If $\alpha \leqslant \beta$, must we have $\alpha+\gamma \leqslant \beta+\gamma$? If $\alpha < \beta$, must we have $\alpha+\gamma < \beta+\gamma$?

4. Show that the inductive and synthetic definitions of ordinal multiplication agree.

5. Is there a non-zero ordinal α with $\alpha\omega = \alpha$? What about $\omega\alpha = \alpha$?

6. Let α, β, γ be ordinals. Show that $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

7. Let α, β, γ be ordinals. Must we have $(\alpha+\beta)\gamma = \alpha\gamma+\beta\gamma$? Must we have $\alpha(\beta+\gamma) = \alpha\beta+\alpha\gamma$?

8. Let α and β be ordinals with $\alpha \geq \beta$. Show that there is a unique ordinal γ such that $\beta+\gamma = \alpha$. Must there exist an ordinal γ with $\gamma+\beta = \alpha$?

9. Show that, for every countable ordinal α, there is a subset of $\mathbb{Q}$ of order-type α. Why is there no subset of $\mathbb{R}$ of order-type ω_1?

10. For well-ordered sets X and Y where X has least element a, let X^{Y*} be the set of functions $f\colon Y \to X$ such that $f(y) = a$ for all but finitely many $y \in Y$. For $f,\, g \in X^{Y*}$ with $f \neq g$, define $f < g$ if $f(b) < g(b)$ where $b = \max\{y \in Y : f(y) \neq g(y)\}$. Show that X^{Y*} is well-ordered by $<$. Show that for ordinals α and β, the order-type of $\alpha^{\beta*}$ is α^β.

11. What is the smallest fixed point of $\alpha \mapsto \omega^\alpha$? The next smallest? And the next smallest? Show that the fixed points are unbounded, and explain why this means that we may index the fixed points by the ordinals. Is there a countable ordinal α such that α is the α-th fixed point?

12. Which of the following posets (ordered by inclusion) are complete?
(i) The set of all subsets of $\mathbb{N}$ that are finite or have finite complement
(ii) The set of all independent subsets of a vector space V
(iii) The set of all subspaces of a vector space V

13. Show that for every set S, the poset $(\mathcal{P}(S), \subset)$ is a Boolean algebra. Show also that every finite Boolean algebra is isomorphic to $(\mathcal{P}(S), \subset)$ for some set S. What happens if we drop the finiteness condition?

14. Let P be a set of primitive propositions and define a relation $\Leftrightarrow$ on $L = L(P)$ by $p \Leftrightarrow q$ iff $p \vdash q$ and $q \vdash p$. Show that $\Leftrightarrow$ is an equivalence relation, that $\vdash$ induces a well-defined relation $\overline{\vdash}$ on $L/\Leftrightarrow$ and that $(L/\Leftrightarrow, \overline{\vdash})$ is a Boolean algebra. Show that a function $v\colon L \to \{0,1\}$ is a valuation iff v is constant on equivalence classes of $\Leftrightarrow$ and the induced function $\overline{v}\colon L/\Leftrightarrow \to \{0,1\}$ is a homomorphism of Boolean algebras.