LINEAR ANALYSIS – EXAMPLES 4

Michaelmas 2019 SBK

1 . For a fixed $(a_n)_{n=1}^{\infty} \in l_\infty$, define the operator $T : l_2 \to l_2$ by $T((x_n)_{n=1}^{\infty}) = (a_n x_n)_{n=1}^{\infty}$. Prove that T is continuous and that $\|T\| = \|a\|_\infty$. Find the eigenvalues (point spectrum) and the spectrum of T. When is T compact?

2 . Let H be a Hilbert space. Let $U \in \mathcal{B}(H)$ be a *unitary* operator on H, which means that U is invertible and $\langle U(x), U(y)\rangle = \langle x, y\rangle$ for all $x, y \in H$. Show that $\sigma(U) \subset \{z \in \mathbf{C} : |z| = 1\}$.

3 . Let K be a non-empty compact subset of $\mathbf{C}$. Show that there is an operator $T : l_2 \to l_2$ with $\sigma(T) = K$.

4 . Let $(a_{ij})_{i,j=1}^{\infty}$ be a complex matrix with $\sum_{i,j=1}^{\infty} |a_{ij}|^2 < \infty$. Show that we can define an operator $T : l_2 \to l_2$ by $T\big(\sum_{i=1}^{\infty} x_i e_i\big) = \sum_{i=1}^{\infty}\big(\sum_{j=1}^{\infty} a_{ij} x_j\big) e_i$. Prove also that T is compact.

5 . Are the invertible operators dense in $\mathcal{B}(l_2)$?

6 . Construct a self-adjoint operator T on a (non-zero) Hilbert space H such that T has no eigenvalues.

7 . Construct a compact operator T on a (non-zero) Hilbert space H such that T has no eigenvalues.

8 . Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Prove that T compact $\Longrightarrow T^*$ compact.

9 . Let X be a Banach space and $T \in \mathcal{B}(X)$ a compact operator. Suppose that $\lambda \in \mathbf{C} \setminus \{0\}$ is not an eigenvalue of T. Show that there is a number $c > 0$ such that $\|T(x) - \lambda x\| \geq c\|x\|$ for every $x \in X$.

10 . Let X be a complex Banach space, $T \in \mathcal{B}(X)$ and let $p : \mathbf{C} \to \mathbf{C}$ be a non-constant complex polynomial. Show that $\sigma(p(T)) = p(\sigma(T))$.