Mich. 2023 **LINEAR ANALYSIS – EXAMPLES 4** IBL

1. For a fixed $a = (a_n) \in l_\infty$, define $T : l_2 \to l_2$ by $T\left(\sum x_n e_n\right) = \sum a_n x_n e_n$. Find the eigenvalues, the approximate eigenvalues and the spectrum of T.

2. Let K be a non-empty compact subset of $\mathbb{C}$. Show that there is an operator T on l_2 whose spectrum is precisely K.

3. Let H be a Hilbert space with an orthonormal basis $(e_n)_{n=-\infty}^{\infty}$. The *bilateral shift* is the operator T on H defined by $T(e_n) = e_{n+1}$. Find the spectrum of the bilateral shift.

4. Let $A = (a_{ij})_{i,j=1}^{\infty}$ be a complex matrix with $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|^2 < \infty$. Show that we can define an operator T on l_2 by $T\left(\sum x_i e_i\right) = \sum_i \left(\sum_j a_{ij} x_j\right) e_i$.

5. Are the invertible operators dense in $L(l_2)$?

6. Let S and T be operators on a Hilbert space H. Must we have $r(S+T) \leq r(S) + r(T)$? Must we have $r(ST) \leq r(S)r(T)$?

7. Let T be an operator on a Hilbert space H. Prove that T compact implies T^* compact.

8. Let T be a unitary operator on a Hilbert space H, and suppose that T acts on a closed subspace Y. Must the restriction of T to Y be unitary?

9. Construct a hermitian operator T on a (non-zero) Hilbert space H such that T has no eigenvalues.

10. Construct a compact operator T on a (non-zero) Hilbert space H such that T has no eigenvalues.

11. Let $A = (a_{ij})_{i,j=1}^{\infty}$ be a complex matrix, and suppose that for some constant k we have $\sum_j |a_{ij}|^2 \leq k$ for all i and $\sum_i |a_{ij}|^2 \leq k$ for all j. Can we define an operator T on l_2 by $T\left(\sum x_i e_i\right) = \sum_i \left(\sum_j a_{ij} x_j\right) e_i$?

12. Let T be an operator on a Banach space X, with $\dim X > 1$. Prove that T has a (not necessarily closed) invariant subspace Y other than $\{0\}$ and X.

13. Let H be a Hilbert space, and let $f : [0,1] \to H$ be a continuous function. Suppose that f is 'everywhere orthogonal', in the sense that $f(x) - f(y) \perp f(y) - f(z)$ for every $x < y < z$. Must f be constant?

$^+$14. Is l_2 homeomorphic to $l_2 - \{0\}$?