Mich. 2023 **LINEAR ANALYSIS – EXAMPLES 3** IBL

1. Let Y be a closed subset of a normal space X, and let $f : Y \to \mathbb{C}$ be a bounded continuous function. Prove that f can be extended to a bounded continuous function $g : X \to \mathbb{C}$, with $\|g\|_\infty = \|f\|_\infty$.

2. Let Y be a closed subset of a normal space X, and let $f : Y \to \mathbb{R}$ be a (possibly unbounded) continuous function. Prove that f can be extended to a continuous function $g : X \to \mathbb{R}$.

3. Let K be a compact Hausdorff space, and let A be a closed subalgebra of $C_{\mathbb{R}}(K)$. Suppose that A separates the points of K, and that for every $x \in K$ there is an $f \in A$ with $f(x) \neq 0$. Prove that $A = C_{\mathbb{R}}(K)$.

4. Let K be a compact Hausdorff space. Find the maximal (proper) closed subalgebras of $C_{\mathbb{R}}(K)$.

5. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 f(x)x^n\,dx = 0$ for every n. Prove that $f = 0$.

6. Let K be a compact metric space. Show that $C(K)$ is separable.

7. Let E be a closed subspace of $C[0,1]$, and suppose that every $f \in E$ is continuously differentiable. Show that the map $f \mapsto f'$ from E to $C[0,1]$ is continuous, and deduce that E is finite-dimensional.

8. Let X be an inner product space, and let $T : X \to X$ be a linear map. Show that $(Tx, Ty) = (x, y)$ for all $x, y \in X$ if and only if $\|Tx\| = \|x\|$ for all $x \in X$.

9. Let X be a complex inner product space, and let $T : X \to X$ be a linear map. Show that if $(Tx, x) = 0$ for all $x \in X$ then $T = 0$. Does the same conclusion hold in the real case?

10. Let $(X, \|.\|)$ be a normed space. Prove that the norm is induced by an inner product if and only if $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in X$.

11. Construct an inner product space X and a closed subspace F of X such that $F \neq X$ but $F^\perp = \{0\}$.

12. Show that the unit ball of l_2 contains an infinite set S such that $\|x - y\| > \sqrt{2}$ for all distinct $x, y \in S$. Can the constant $\sqrt{2}$ be improved?

$^+$13. Let K be a compact Hausdorff space. Does K separable imply $C(K)$ separable?

$^+$14. Construct a Hausdorff space (of more than 1 point) on which every continuous real-valued function is constant.