Mich. 2023 **LINEAR ANALYSIS – EXAMPLES 2** IBL

1. For a fixed $a = (a_n) \in l_\infty$, define $T : l_2 \to l_2$ by $T\left(\sum x_n e_n\right) = \sum a_n x_n e_n$. Prove that T is continuous, and that $\|T\| = \|a\|_\infty$.

2. A linear functional T on l_∞ is called *positive* if $T(y) \geq 0$ whenever $y_n \geq 0$ for all n. Prove that a positive linear functional on l_∞ is continuous.

3. Prove carefully that ${c_0}^*$ is isometrically isomorphic to l_1 and that ${l_1}^*$ is isometrically isomorphic to l_∞.

4. Let T be a linear functional on a normed space X. Prove that if T is continuous then $\operatorname{Ker} T$ is closed in X, while if T is discontinuous then $\operatorname{Ker} T$ is dense in X.

5. For which $a \in l_\infty$ is the operator T in Question 1 compact?

6. Let X and Y be normed spaces that are dense in Banach spaces $\widetilde{X}$ and $\widetilde{Y}$ respectively, and let $T \in L(X, Y)$. Explain why T extends to a unique $\widetilde{T} \in L(\widetilde{X}, \widetilde{Y})$. Show that $\left\|\widetilde{T}\right\| = \|T\|$, so that we may regard $L(X, Y)$ as a subspace of $L(\widetilde{X}, \widetilde{Y})$. Is $L(X, Y)$ dense in $L(\widetilde{X}, \widetilde{Y})$? If T is surjective, must $\widetilde{T}$ be surjective? If T is injective, must $\widetilde{T}$ be injective?

7. Does there exist a discontinuous linear map on a Banach space?

8. Let X be a (non-empty) countable complete metric space. Prove that X has an isolated point.

9. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function such that for every $x > 0$ we have $f(nx) \to 0$ as $n \to \infty$. Show that $f(x) \to 0$ as $x \to \infty$.

10. Let X be a closed subspace of $C[0, 1]$. Suppose that for every $f \in C[0, 1/2]$ there exists $g \in X$ whose restriction to $[0, 1/2]$ is f. Show that there is a constant c such that the function g may always be chosen to satisfy $\|g\| \leq c\|f\|$.

11. Let $\|.\|$ be a complete norm on $C[0, 1]$ such that, for every $x \in [0, 1]$, the evaluation map $f \mapsto f(x)$ is continuous. Prove that $\|.\|$ is equivalent to the uniform norm.

12. Let $T : l_2 \to l_2$ be a linear map such that, for every $y \in l_2$, the map $x \mapsto T(x).y$ is continuous. Must T be continuous?

13. Does there exist a complete norm on F, the space of finite sequences?

$^+$14. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function such that for every $x \in \mathbb{R}$ there is an n with $f^{(m)}(x) = 0$ for all $m \geq n$. Prove that f is a polynomial.