LINEAR ANALYSIS – EXAMPLES 4

H is a complex Hilbert space and $\mathcal{B}(H)$ the bounded linear operators from H to H.

1. Let $T \in \mathcal{B}(H)$ with $|||T||| < 1$. Show that there is $R \in \mathcal{B}(H)$ such that $R^2 = 1 - T$.

2. Let $T \in \mathcal{B}(H)$. Give a definition of $f(T)$ for a rational function f with no poles in $\sigma(T)$, and prove that $\sigma(f(T)) = \{f(\lambda) : \lambda \in \sigma(T)\}$.

3. Let F be a closed subspace of H. Show that $F^{\perp\perp} = F$. Deduce that if $S \subset H$ then $S^{\perp\perp} = \overline{\text{span}(S)}$, and that S has dense linear span in H iff $S^\perp = \{0\}$.

4. Let (a_n) a bounded sequence in $\mathbb{C}$ and $T : \ell^2 \to \ell^2$, $(x_n) \mapsto (a_n x_n)$. Show that T is bounded with $|||T||| = \|(a_n)\|_\infty$. Find the eigenvalues, approximate eigenvalues and the continuous and residual spectrum of T. Show that T is compact if and only if $a_n \to 0$.

5. Show that $T \in \mathcal{B}(H)$ is normal iff $\|Tv\| = \|T^* v\|$ for all $v \in H$.

6. Let $U \in \mathcal{B}(H)$ be a unitary operator. Show that $\sigma(U) \subset \mathbb{S}^1$.

7. Assuming H to be infinite-dimensional, are invertible operators dense in $\mathcal{B}(H)$?

8. Construct $S \in \mathcal{B}(H)$ self-adjoint with no eigenvalues, and, when H is separable, construct $C \in \mathcal{B}(H)$ compact with no eigenvalues. Is the latter possible if H not separable?

9. Assuming $(e_n)_{n \in \mathbb{Z}}$ to be a Hilbert basis, the *bilateral shift operator* is defined by $Te_n = e_{n+1}$. Find the spectrum of T.

10. Let $T \in \mathcal{B}(H)$ compact, show that $T^* \in \mathcal{B}(H)$ is compact.

11. Let $T \in \mathcal{B}(B)$ compact and $\lambda \notin (\sigma_p(T) \cup \{0\})$. Show that $T - \lambda$ is bounded below.

12. Let $T \in \mathcal{B}(H)$ compact self-adjoint. For any $\lambda \in \mathbb{R} \setminus \{0\}$, show that: (a) Either the only solution to $Tx = \lambda x$ is $x = 0$ and $T - \lambda$ is invertible, (b) or $N_\lambda := \text{Ker}(T - \lambda) \neq \{0\}$ finite-dimensional, and given any $x_0 \in H$ the equation $Tx = \lambda x + x_0$ has a solution $x \in H$ iff $x_0 \perp N_\lambda$ (and the space of solutions has $\dim N_\lambda$).

13. Let $U \in \mathcal{B}(H)$ unitary. Show that for all $x \in H$, the sequence $n^{-1} \sum_{k=0}^{n-1} U^k(x)$ converges to the orthogonal projection of x onto the closed subspace $F := \text{Ker}(U - \text{Id})$.

***14.** Given $T \in \mathcal{B}(H)$, define $W(T) := \{\langle Tx, x\rangle,\ \|x\| = 1\} \subset \mathbb{C}$. Show that $W(T)$ is convex and $\sigma(T) \subset \overline{W(T)}$. If T self-adjoint, show that $\overline{W(T)}$ is the convex hull of $\sigma(T)$.

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