LINEAR ANALYSIS – EXAMPLES 2

1. Let $f : V \to \mathbb{R}$ linear with V NVS. Prove that f is continuous iff $\ker f := f^{-1}(\{0\})$ is closed. When f is discontinuous prove that $\ker f$ is dense in V.

2. Given $(f_i)_{i\in I}$ an arbitrary collection of continuous functions $[0,1] \to \mathbb{R}$ such that $\sup_{i\in I} |f_i(x)| < +\infty$ at each $x \in [0,1]$, show that there is an interval $[a,b] \subset [0,1]$ with $a < b$ such that $\sup_{x\in[a,b]} \sup_{i\in I} |f_i(x)| < +\infty$.

3. Let X be a closed subspace of ℓ^1. Assume that for every $(y_n) = (x_{2n}) \in \ell^1$ there exists an extension $(x_n) \in X$ (adding the odd indices). Show that there is $C > 0$ such that for any $(y_n) = (x_{2n}) \in \ell^1$, there is an extension $(x_n) \in X$ with $\|(x_n)\|_{\ell^1} \leq C\|(y_n)\|_{\ell^1}$.

4. Let V Banach space, W a NVS and $T : V \to W$ a bounded linear map. Assume that there are $M > 0$ and $\alpha \in (0,1)$ such that for any $w \in \overline{B}_W(0,1)$ (closed unit ball of W) there is $v \in \overline{B}_V(0,M)$ such that $\|Tv - w\|_W \leq \alpha$. Prove successively that $T(\overline{B}_V(0, \frac{M}{1-\alpha})) \supset \overline{B}_W(0,1)$, that T is surjective and open, and that W is complete.

5. Assume that W is a closed subspace of $V := (C[0,1], \|\cdot\|_\infty)$ that is included in $C^1([0,1])$. Show that W is finite-dimensional. [*You can use the Arzelà-Ascoli theorem.*]

6. Can a proper subspace of a Banach space be non-meagre (of second category)?

7. Given $f : \mathbb{R} \to \mathbb{R}$ continuous such that $f(nx) \xrightarrow{n\to+\infty} 0$ for all $x > 0$, prove that $f(x) \to 0$ as $x \to +\infty$.

8. Given a sequence $f_n \in C([0,1])$ and $f : [0,1] \to \mathbb{R}$ such that $f_n(x) \to f(x)$ for each $x \in [0,1]$, prove that the set of points where f is discontinuous is meagre. Can the Dirichlet function $\mathbf{1}_{\mathbb{Q}}$ be a derivative?

9. Does there exist $f : \mathbb{R} \to \mathbb{R}$ continuous at any $x \in \mathbb{Q}$ and discontinuous elsewhere?

10. Is there a norm $\|\cdot\|$ on c_{00}, the vector space of sequences which have only finitely many nonzero elements, so that $(c_{00}, \|\cdot\|)$ is a Banach space?

11. Let V, W normed vector spaces with V complete. Given $T : V \to W$ a bounded linear map, $v \in V$ and $r > 0$, prove that $\sup_{v'\in B(v,r)} \|Tv'\| \geq |||T|||r$. Given $\mathcal{F} = (T_i)_{i\in I}$ set of bounded linear maps such that $(T_i(v))_{i\in I}$ bounded for each $v \in V$ and $(|||T_i|||)_{i\in I}$ not bounded, construct sequences T_n in $\mathcal{F}$ and $v_n \in V$ such that $\|v_n - v_{n-1}\| \leq 3^{-n}$ and $\|T_n v_n\| \geq (2/3)3^{-n}|||T_n|||$ and $|||T_n||| \geq 4^n$. Deduce another proof of the uniform boundedness principle.

***12.** Prove that a NVS homeomorphic to a complete metric space is a Banach space.

***13.** Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function such that for every $x \in \mathbb{R}$ there is an $n \geq 1$ such that $f^{(n')}(x) = 0$ for all $n' \geq n$. Prove that f is a polynomial.

Date: Michaelmas 2022 – For comments, email `C.Mouhot@dpmms.cam.ac.uk`.