LINEAR ANALYSIS – EXAMPLES 1

1. Given $p \in [1, +\infty]$, show that $\|x\|_p := (\sum |x_n|^p)^{\frac{1}{p}}$ is a norm on ℓ^p (prove in particular carefully the triangle inequality), and show that $(\ell^p, \|\cdot\|_p)$ is a Banach space.

2. For which NVS does there exist a discontinuous linear map from the NVS to itself?

3. Given W_1, W_2 dense subspaces of a NVS V, is $W_1 \cap W_2$ dense in V?

4. Prove that $C([0,1])$ endowed with the norm $\|f\|_\infty = \sup_{[0,1]} |f|$ is separable (i.e. has a countable dense subset) whereas ℓ^∞ endowed with the norm $\|(x_n)\| = \sup_{n \geq 1} |x_n|$ isn't.

5. Let V infinite-dimensional NVS, show that there is a sequence (x_n) in V with $\|x_n\| \leq 1$ and $\|x_m - x_n\| \geq 1$ whenever $m \neq n$. Can the closed unit ball of V be compact?

6. Let V be a NVS and $\pi : V \setminus 0 \to V$ defined by $\pi(v) = \frac{v}{\|v\|}$. Is it always true that $\|\pi(v) - \pi(w)\| \leq \|v - w\|$ whenever $\|v\|, \|w\| \geq 1$?

7. Prove that a NVS is complete iff every series $\sum_{n \geq 1} x_n$ with $\sum_{n \geq 1} \|x_n\| < +\infty$ is convergent.

8. Show that the space $C^1([0,1])$ is incomplete in the norm $\|f\|_\infty = \sup_{[0,1]} |f|$ but complete in the norm $\|f\|_\infty + \|f'\|_\infty$.

9. Prove that (i) for $1 < p < q < \infty$ and $x \in \mathbb{C}^n$ the inequality $\|x\|_q \leq \|x\|_p \leq n^{\frac{1}{p} - \frac{1}{q}} \|x\|_q$ holds and cannot be improved, (ii) for $p, q \in [1, \infty]$, the inclusion $\ell^p \subset \ell^q$ holds iff $p \leq q$, (iii) when the latter inclusion holds it is continuous.

10. Given $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$, prove that $(\ell^p)^*$ and ℓ^q are isometrically isomorphic (usually simply denoted $(\ell^p)^* = \ell^q$).

11. Let c_0 the space of sequences that converge to zero and c the space of sequences that converge (to any limit). Prove that they are complete in the norm ℓ^∞, that $(c_0)^* = \ell^1$ and that c_0 and c are isomorphic as TVS (i.e. there is a linear homeomorphism between them). Is $\cup_{p \in [1,\infty)} \ell^p = c_0$?

***12.** Prove that $(\ell^1)^* = \ell^\infty$. Consider $\mathbf{x}^k = (x_n^k)$ a sequence of sequences $\mathbf{x}^k \in \ell^1$ such that $\sum_{n \geq 1} x_n^k y_n \to 0$ as $k \to \infty$ for any $(y_n) \in \ell^\infty$. Prove that $\mathbf{x}^k \to 0$ in ℓ^1.

***13.** Let $X_p := \ell^p$ when $p \in [1, +\infty)$ and $X_\infty := c_0$. Given $1 \leq p < q \leq \infty$, prove that every bounded linear operator from X_q to X_p maps the unit ball into a relatively compact set. Show that no two of the spaces $\{\ell^p,\ p \in [1, +\infty]\}$, c_0 are isomorphic.

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