LINEAR ANALYSIS – EXAMPLES 3

The "*" questions are *strictly optional* given the higher number of normal questions.

1. Let X normal topological space and $S \subset X$. Show that there is a continuous function $f : X \to \mathbb{R}$ such that $S = f^{-1}(\{0\})$ iff S is a closed countable intersection of open sets.

2. Given K Hausdorff compact, prove: $C(K)$ is finite-dimensional iff K is a finite set.

3. Given K Hausdorff compact and a finite open cover $K \subset \cup_{i=1}^n U_i$, show that there are continuous functions $\chi_i : K \to [0,1]$ such that $\chi_i = 0$ on $K \setminus U_i$ and $\sum_{i=1}^n \chi_i = 1$ on K.

4. Given K Hausdorff compact, show that $C(K)$ is separable iff K is metrizable.

5. Let X *separable* Hausdorff compact space and (f_n) equi-bounded and equi-continuous on X. Given Y countable dense in X prove, by a diagonal argument, that a subsequence of (f_n) converges pointwise on Y. Deduce a proof of the Arzelà-Ascoli Theorem.

6. Given K Hausdorff compact, prove, using the Arzelà-Ascoli Theorem or otherwise, that if $f_n \in C_{\mathbb{R}}(K)$ is such that $f_{n+1}(x) - f_n(x)$ has constant sign for all $x \in K$ and $n \geq 1$ and f_n converges pointwise to a continuous limit, then (f_n) converges uniformly.

7. Consider the product rule $(f * g)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x-y) g(y) \, \mathrm{d}y$ on $C(\mathbb{T})$. Prove that $(C(\mathbb{T}), \| \cdot \|_\infty)$ is a Banach Algebra with this product, that is not unitary.

8. Consider $f \in C([0,1])$ such that $\int_0^1 f(x) x^n \, \mathrm{d}x = 0$ for all $n \geq 0$. Prove that $f = 0$.

9. Given $f \in C(\mathbb{T})$ and $S_N(x) := \sum_{k=-N}^{+N} \hat{f}(k) e^{ikn}$ prove the formula:

$$G_N := \frac{1}{N} \sum_{\ell=0}^{N-1} S_\ell = (F_N * f) \quad \text{with} \quad F_N(x) := \sum_{\ell=-N}^{+N} \left(1 - \frac{|\ell|}{N}\right) e^{i\ell x} = \frac{1}{N} \left(\frac{\sin(\frac{Nx}{2})}{\sin(\frac{x}{2})} \right)^2.$$

Prove that $G_N \to f$ uniformly and deduce an alternative proof the Weierstrass approximation Theorem (i.e. the density of polyomials in $C([0,1])$ for the uniform convergence).

10. Let $T : E \to E$ linear isometry on E Euclidean, show $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all x, y.

11. Let V be a complex inner product space and $T : V \to V$ a linear map. Show that if $\langle Tx, x \rangle = 0$ for all $x \in V$, then $T = 0$. Does the same conclusion hold in the real case?

12. Given H separable Hilbert space, are there F_1 and F_2 two closed subspaces different from H so that $F_1 \cap F_2 = \{0\}$ and $F_1 + F_2$ dense in H but not H?

13. Show that the unit ball of ℓ^2 contains sequences (x_n) so that $\|x_m - x_n\| > \sqrt{2}$ for all $m, n \geq 1$ so that $m \neq n$. Can the constant $\sqrt{2}$ be improved (made bigger)?

14. Construct E Euclidean space and $F \subset E$ closed such that $F \neq E$ and $F^\perp = \{0\}$.

15. Compute $\sum_{n \geq 1} \frac{1}{n^4}$ with the Parseval identity applied to a suitable $f \in C(\mathbb{T})$.

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***16.** Is there a continuous surjective map $\mathbb{R} \to \ell^2$?

***17.** Given X a non-empty set, we say that $a : X \to \mathbb{R}_+$ is *summable* if there is $M \geq 0$ so that $\sum_{x \in F} a(x) \leq M$ for any finite subset $F \subset X$, and we define $\ell^2(X)$ the maps $b : X \to \mathbb{C}$ so that $|b|^2$ is summable. Prove that summability implies that the support is countable. Prove that $\ell^2(X)$ is a Hilbert space, which is separable iff X is countable.

***18.** Let X be a topological space such every open cover has a countable sub-cover, and such that for every closed subset F and $x \notin F$ there are disjoint open sets U and V with $x \in U$ and $F \subset V$. Prove that X is normal.

***19.** Consider the vector space of the trigonometric polynomials on $\mathbb{R}$. Prove that it is a Euclidean space when endowed with the inner product

$$\langle f, g \rangle := \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{+T} f(x)\overline{g(x)}\,\mathrm{d}x.$$

Use it to build a *non-separable* Hilbert space H, for which there exists $f : [0,1] \to H$ continuous so that for every $x < y < z$ in $[0,1]$ we have $f(x) - f(y) \perp f(y) - f(z)$.