LINEAR ANALYSIS – EXAMPLES 3

Michaelmas 2019 SBK

1 . Let Y be a closed subset of a normal topological space X and let $f : Y \to \mathbf{C}$ be a bounded continuous function. Prove that f can be extended to a bounded continuous function $g : X \to \mathbf{C}$ with $g\big|_Y = f$ and $\|g\|_\infty = \|f\|_\infty$.

2 . Let Y be a closed subset of a normal topological space X and let $f : Y \to \mathbf{R}$ be a possibly unbounded continuous function. Prove that f can be extended to a continuous function $g : X \to \mathbf{R}$ with $g\big|_Y = f$.

3 . Let K be a compact, Hausdorff space. For any cover of K by open sets $U_1, \dots U_n$, show that there exists a *partition of unity* subordinate to the cover $\{U_i\}_{i=1}^n$, *i.e.* show that there exist continuous functions $\varphi_i : K \to [0,1]$ such that $\varphi_i(x) = 0$ for $x \notin U_i$ and $\sum_{i=1}^n \varphi_i(x) = 1$ for every $x \in K$.

4 . Let K be a compact, Hausdorff space and let $A \subset C_{\mathbf{R}}(K)$ be a closed subalgebra. Suppose that A separates the points of K and that for every $x \in K$, there is an $f \in A$ with $f(x) \neq 0$. Show that $A = C_{\mathbf{R}}(K)$.

5 . Let $f : [0,1] \to \mathbf{R}$ be a continuous function such that $\int_0^1 f(x)x^n \, dx = 0$ for every $n \geq 0$. Show that $f = 0$.

6 . Let K be a compact metric space. Show that $C(K)$ is separable.

7 . Let X be an inner product space and let $T : X \to X$ be a linear map. Show that $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for every $x, y \in X$ if and only if $\|T(x)\| = \|x\|$ for all $x \in X$.

8 . Let $(X, \|\cdot\|)$ be a normed space. Prove that the norm is induced by an inner product if and only if the parallelogram identity holds:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \text{for all } x, y \in X.$$

9 . Construct an inner product space X and a closed subspace $F \subset X$ such that $F \neq X$ but $F^\perp = \{0\}$.

10 . Show that the unit ball of l_2 contains an infinite set S with $\|x - y\| > \sqrt{2}$ for every distinct $x, y \in S$. Can the constant $\sqrt{2}$ be improved (made bigger)?

$^+$11. Is there a Hausdorff space containing more than one point on which every continuous, real-valued function is constant?