LINEAR ANALYSIS – EXAMPLES 2

Michaelmas 2019 SBK

1 . *(Completion of Normed Spaces)* Let X be a normed space and let $\widetilde{X}$ be the set of all Cauchy sequences $(x_n)_{n=1}^{\infty}$ in X quotiented by the equivalence relation $(x_n)_{n=1}^{\infty} \sim (y_n)_{n=1}^{\infty} \Leftrightarrow \|x_n - y_n\| \to 0$. Write $[(x_n)]$ for the equivalence class of $(x_n)_{n=1}^{\infty}$ and make $\widetilde{X}$ into a vector space by defining $\lambda[(x_n)] + \mu[(y_n)] = [(\lambda x_n + \mu y_n)]$. Show that the norm on $\widetilde{X}$ defined by $\|[(x_n)]\| := \lim_{n\to\infty} \|x_n\|$ is complete and that X is isometrically isomorphic to a dense subset of $\tilde{X}$.

2 . Prove carefully that c_0^* is isometrically isomorphic to l_1 and that l_1^* is isometrically isomorphic to l_∞.

3 . A linear functional T on l_∞ is said to be *positive* if $T(y) \geq 0$ whenever $y_n \geq 0$ for all n. Prove that a positive linear functional on l_∞ is continuous.

4 . Let T be a linear functional on a normed space X. Prove that if T is continuous then $\ker T$ is closed in X, while if T is discontinuous then $\ker T$ is dense in X.

5 . Let X be a (non-empty) countable complete metric space. Prove that X has an isolated point (a point $x \in X$ such that $\{x\}$ is open).

6 . Let $(f_n)_{n=1}^{\infty} \in C([0,1])$ and suppose that for each $x \in [0,1]$, we have $\sup_n |f_n(x)| < \infty$. Show that there is a (non-empty) open subinterval $(a,b) \subset [0,1]$ for which $\sup_n \sup_{x\in(a,b)} |f_n(x)| < \infty$.

7 . Let $1 < p, q < \infty$ with $1/p + 1/q = 1$, and let $y = (y_n)$ be a sequence of reals. Show that if $\sum_{n=1}^{\infty} x_n y_n$ converges for every $x \in l_p$, then $y \in l_q$.

8 . The 'L^p-norm' on $C([0,1])$ is $\|f\|_p := \left(\int_0^1 |f(x)|^p \, dx\right)^{1/p}$. Show that $\|f\|_p \leq \|f\|_\infty$. Deduce that $\|\cdot\|_p$ is not complete on $C([0,1])$.
(The completion - in the sense of Q1. - is the space $L^p([0,1])$.)

9 . Let $\|\cdot\|$ be a complete norm on $C([0,1])$ such that for each $x \in [0,1]$, the functional $f \mapsto f(x)$ is continuous. Show that $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$.

10 . Does there exist a complete norm on c_{00}?

11 . Let $f : (0,\infty) \to \mathbf{R}$ be a continuous function such that for any $x > 0$ we have $f(nx) \to 0$ as $n \to \infty$ as $n \to \infty$. Show that $f(x) \to 0$ as $x \to \infty$.