LINEAR ANALYSIS – EXAMPLES 1

Michaelmas 2019 SBK

1 . Prove that a normed space X is a Banach space if and only if every series $\sum_{n=1}^{\infty} x_n$ in X with $\sum_{n=1}^{\infty} \|x_n\| < \infty$ is convergent.

2 . Prove that there does not exist a finite basis for l_1.

3 . Show directly that the spaces l_p for $1 \leq p \leq \infty$, and the space c_0 are complete.

4 . Let $1 \leq p < q < \infty$. Show that the set l_p is a subset of $(l_q, \|\cdot\|_q)$ with empty interior. Is the set l_p closed in $(l_q, \|\cdot\|_q)$?

5 . Show that $C^1([0,1]) = \big\{ f \in C([0,1]) : f \text{ is continuously differentiable} \big\}$ is not complete with the uniform norm $\|\cdot\|_\infty$, and show that it is complete with the norm $\|f\| := \|f\|_\infty + \|f'\|_\infty$.

6 . Let $1 < p < \infty$ and let x and y be vectors in l_p with $\|x\| = \|y\| = 1$ and $\|x + y\| = 2$. Prove that $x = y$.

7 . Let $1 < p, q, r < \infty$ be such that $1/p + 1/q + 1/r = 1$. Show that if $x \in l_p$, $y \in l_q$ and $z \in l_r$, then $\sum_{n=1}^{\infty} |x_n y_n z_n| \leq \|x\|_p \|y\|_q \|z\|_r$.

8 . Let Y and Z be dense subspaces of the normed space X. Must $Y \cap Z$ be dense in X?

9 . Give an example of a vector space V and two inequivalent norms $\|\cdot\|_A$ and $\|\cdot\|_B$ on V such that the normed spaces $(V, \|\cdot\|_A)$ and $(V, \|\cdot\|_B)$ are isomorphic.

10 . Let x and y be vectors in a normed space X with $\|x\|, \|y\| \geq 1$. Is it always true that

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \|x - y\| \,?$$

11 . Let Y be a proper, closed subspace of a normed space X. Is there always a non-zero vector $x \in X$ such that $\|x + y\| \geq \|y\|$ for all $y \in Y$?

12 . Prove that c_0 and l_p for $1 \leq p < \infty$ are separable and prove that l_∞ is not separable.

$^+$13. Is l_2 homeomorphic to $l_2 \setminus \{0\}$?