

1. Let $1 \leq p < q < \infty$. Show that $\ell_p \subsetneq \ell_q \subsetneq c_0$. Is $\bigcup_{p \in [1, \infty)} \ell_p = c_0$?
2. Show directly that the spaces ℓ_p , $1 \leq p \leq \infty$, are complete.
3. Show that the space $C^1[0, 1] = \{f \in C[0, 1] : f \text{ continuously differentiable}\}$ is incomplete in the uniform norm $\|\cdot\|_\infty$ but complete in the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$.
4. Show carefully that $\ell_1^* \cong \ell_\infty$ and $c_0^* \cong \ell_1$.
5. Prove that ℓ_∞ is not separable.
6. Prove that a normed space is a Banach space if and only if every series $\sum x_n$ in X with $\sum \|x_n\| < \infty$ is convergent.
7. Let $T: \ell_p^n \rightarrow \ell_q^n$ be the identity map of the underlying vector space \mathbb{R}^n . Compute the operator norm of T for all possible values of p and q .
8. Let X be a normed space. For $x \in X \setminus \{0\}$ write $\pi(x) = x/\|x\|$. Is it true that $\|\pi(x) - \pi(y)\| \leq \|x - y\|$ whenever $\|x\|, \|y\| \geq 1$?
9. Show that the spaces c_0 and c are isomorphic.
10. Show that no two of the spaces $\ell_1, \ell_2, \ell_\infty, c_0$ are isomorphic.
11. Let Y and Z be dense subspaces of a normed space X . Is $Y \cap Z$ dense in X ?
12. Assume that X is an infinite-dimensional normed space. Show that there is a sequence (x_n) in the unit ball of X with $\|x_m - x_n\| \geq 1$ whenever $m \neq n$. Is it possible to replace \geq by $>$?
13. Let Y be a proper closed subspace of a normed space X . Given $x \in X$, must there be a $y \in Y$ such that $d(x, Y) = \|x - y\|$? Also, must there exist $x \in B_X$ such that $d(x, Y) = 1$?
14. Does there exist a discontinuous linear map on a Banach space?
15. Show that there exist two non-equivalent complete norms on a vector space.
- +16. Construct two normed spaces X, Y such that $d(X, Y) = 1$ but X and Y are not isometrically isomorphic.