

- Let  $(X, d_X)$ ,  $(Y, d_Y)$  be metric spaces. Define  $d$  on  $X \times Y$  by  $d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$ . Show that  $d$  makes  $X \times Y$  into a metric space.
- Show that if  $\phi : X \rightarrow Y$  is continuous, then the graph  $\Gamma$  of  $\phi$ , defined by  $\Gamma = \{(x, \phi(x))\}$ , is closed.
- Give an example of metric spaces  $X$  and  $Y$ , and a map  $\phi : X \rightarrow Y$ , such that  $\phi$  is not continuous, yet the graph of  $\phi$  is closed and bounded.
- Let  $X$  be a metric space. Show that the following are equivalent:  $X$  is compact,  $X$  is countably compact,  $X$  is sequentially compact. ( $X$  is countably compact if every countable open cover has a finite subcover.)
- Let  $X$  be a normed vector space, and  $Y$  a *closed* proper subspace. Show that for all  $\epsilon > 0$ , there exists an  $x \in X$ , such that  $\|x\| = 1$ , and

$$\inf_{y \in Y} \|x - y\| \geq 1 - \epsilon$$

- Use the above two exercises to give another proof that if  $X$  is a normed vector space and  $\{x : \|x\| \leq 1\}$  is compact, then  $X$  is finite dimensional.
- Use the Hahn-Banach theorem to show that  $(l_\infty)^* \neq l_1$ .
- Use a category argument to show problem 15 from Example Sheet 1.
- Let  $X$  be a topological space. Show that the space  $C(X)$  of complex-valued bounded continuous functions is a Banach space, with the sup norm  $\|f\| = \sup_{x \in X} |f(x)|$ .
- Prove the following complex version of the Tietze-Urysohn extension theorem: Let  $X$  be a normal topological space, and let  $A \subset X$  be closed. Suppose  $f : A \rightarrow \mathbf{C}$  is continuous with  $\|f\| < \infty$ . Then there exists a continuous function  $\tilde{f} : X \rightarrow \mathbf{C}$  such that  $\tilde{f}|_A = f$ , and  $\|\tilde{f}\| = \|f\|$ .
- Show that there exist continuous nowhere differentiable functions  $f : [0, 1] \rightarrow \mathbf{R}$ . *Hint:* Exhibit the set of nowhere differentiable functions as a countable intersection of dense open sets in  $C[0, 1]$ .
- Let  $T$  denote the unit circle in the usual 2-plane, and let  $C(T)$  denote the space of complex-valued continuous functions on  $T$ . Since  $T$  is compact, by Exercise 9,  $C(T)$  is a Banach space. We will parametrize  $T$  by the angular variable  $t \in [-\pi, \pi]$ .<sup>1</sup> Define,

$$D_n(t) = \sum_{k=-n}^n e^{ikt}$$

and define the operator

$$S_n : C(T) \rightarrow C(T) \tag{1}$$

by

$$(S_n(f))(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_n(x - t) dt.$$

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<sup>1</sup>i.e., one can think of  $C(T)$  as the space of continuous periodic functions on  $\mathbf{R}$  with period  $2\pi$

The function  $S_n(f)$  is called the  $n$ 'th partial sum of the Fourier series of  $f$ , i.e.

$$S_n(f) = \sum_{k=-n}^n \hat{f}(k) e^{ikx},$$

where for  $k \in \mathbf{Z}$ ,

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt. \quad (2)$$

In the next few exercises, we shall examine the question of whether this series necessarily converges to  $f$ .

In this exercise, show that  $S_n$  is indeed a linear operator with target as given by (1). Show that each  $\|S_n\| < \infty$ . Let  $\phi_n$  denote the composition of  $S_n$  with the evaluation-at-0 map  $e_0 : C(T) \rightarrow \mathbf{C}$  given by  $e_0(f) = f(0)$ . Show that  $\|\phi_n\| < \infty$ . Thus  $\phi_n \in C(T)^*$ .

13. Show that in fact,

$$D_n(t) = \frac{\sin(n + \frac{1}{2})t}{\sin(t/2)}.$$

14. Use this to show that  $\|D_n\| \rightarrow \infty$  as  $n \rightarrow \infty$ . Deduce that  $\|\phi_n\| \rightarrow \infty$ . Using the Banach-Steinhaus Theorem, deduce that there exists a function  $f \in C(T)$  whose Fourier series diverges at 0.

15. Show that in fact, there exists a dense set  $\mathcal{F} \subset C(T)$ , such that for each  $f \in \mathcal{F}$ , there exists a dense set  $S_f \subset (-\pi, \pi)$  such that the Fourier series of  $f$  diverges everywhere on  $S_f$ .

16. Let  $f \in C(T)$ . Show that  $\hat{f}(n) \rightarrow 0$ , where  $\hat{f}$  is defined by (2). Thus,  $f \rightarrow \hat{f}$  defines a linear map  $\Lambda : C(T) \rightarrow c_0$ , where  $c_0$  is defined<sup>2</sup> as the Banach space of functions  $\mathbf{Z} \rightarrow \mathbf{C}$ , with norm given by

$$\|\hat{g}\|_\infty = \sup_{n \in \mathbf{Z}} |\hat{g}(n)|.$$

17. For those who know about Lebesgue measure: Define the space  $L^1(T)$  as the set of all Lebesgue integrable functions on  $T$ , modulo null functions. Define

$$\|f\|_1 = \int_{-\pi}^{\pi} |f(t)| dt.$$

This norm makes  $L^1$  into a Banach space. Show that  $\Lambda$  defined by  $f \rightarrow \hat{f}$ , where again  $\hat{f}$  is defined by (2), taken now in the sense of Lebesgue, maps

$$\Lambda : L^1 \rightarrow c_0.$$

Show moreover that this map is bounded and injective.

18. Show that  $D_n(t)$  defined previously satisfies  $\|D_n\|_1 \rightarrow \infty$ , while on the other hand  $\|\Lambda(D_n)\|_\infty = 1$ . Deduce from the inverse mapping theorem that  $T$  is *not surjective*.

For comments, email [M.Dafermos@dpmms.cam.ac.uk](mailto:M.Dafermos@dpmms.cam.ac.uk)

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<sup>2</sup>Compare with the definition in Example Sheet 1.