1. Let X be a normed vector space, and let $T : X \to X$, $S : X \to X$ be bounded linear maps. Show that $T \circ S$ is bounded with

$$||T \circ S|| \leq ||T|| \cdot ||S||.$$

Show by specific example that equality need not hold above.

2. Give an example of Banach spaces X and Y, and a linear map $T : X \to Y$ such that $||Tx|| \leq ||T||$ for all $||x|| \leq 1$.

3. Let $T : X \to Y$ and $S : Y \to Z$ be bounded linear maps. Show that $(S \circ T)^* = T^* \circ S^*$.

4. Let $p > 0$, and define the space l_p of all complex sequences such that $\sum_i |x_i|^p < \infty$. Define

$$||x||_p = \left(\sum_i |x_i|^p\right)^{\frac{1}{p}}. \tag{1}$$

For $1 > p > 0$, is this a vector space? Again, for $1 > p > 0$, is this a normed vector space with norm defined by (1)?

5. Define the space l_∞ consisting of all sequences $\{x_i\}$ such that $\sup |x_i| < \infty$, with

$$||\{x_i\}||_\infty = \sup |x_i|.$$

Show that this defines a Banach space. Recall that a metric space is said to be separable if there exists a countable dense set. Show that l_∞ is not separable.

6. Let $T : V \to W$ be a linear map between finite dimensional normed vector spaces, let e_i denote a basis for V, let $\hat{e}_j$ denote a basis for W, and let a_{ij} denote the components of the matrix representing T in this basis. Determine a basis for W^* and V^* for which T^* has a nice form, and give that form.

7. Let $0 \leq t \leq 1$. Let a and b be nonnegative real numbers. Prove $a^t b^{1-t} \leq ta + (1-t)b$.

8. Let $p \geq 1$, and define q by the relation $p^{-1} + q^{-1} = 1$, with the convention that if $p = 1$, $q = \infty$. We call p and q conjugate exponents. Show that if $p > 1$, and if $\{x_i\}$ and $\{y_i\}$ are elements of l_p and l_q, respectively, then $\{x_i y_i\}$ is in l_1 and *Hölder's inequality* holds, i.e.

$$||x_i y_i||_1 \leq ||x_i||_p ||y_i||_q.$$

9. Show that l_p, for all $p \geq 1$, is a Banach space with norm defined by (1), i.e. show that l_p is a vector space, that (1) defines a norm, and that the induces metric space is complete.

10. Show that for $1 \leq p < \infty$, $l_p^* = l_q$.

11. Denote by c_0 the subset of l_∞, consisting of all sequences tending to 0. Show that $c_0^* = l_1$.

12. Let X be a vector space, and let $\{p_i\}$ be a countable collection of seminorms, such that for all $x \in X$, there exists an i such that $p_i(x) > 0$. Fix $1 \leq p \leq \infty$, and define

$$||x|| = ||\{p_i(x)\}||_p,$$

where the right hand side denotes the l_p norm. Let Y denote the subset of X consisting of all x such that the above is finite. Does $||\cdot||$ endow Y with the structure of a normed vector space?

13. Let X be a normed vector space such that its dual is reflexive. Show that X itself is reflexive.

14. Recall the space l_2^n, i.e. the normed vector space $\mathbf{C}^n$ with norm defined by

$$|(x_1, \ldots, x_n)| = \sqrt{\sum_{i=1}^{n} |x_i|^2}.$$

Let $T : X \to X$ be a linear map. Describe $||T||$ algebraically.

15. Let V be a vector space with a countable basis. Show that V cannot be made into a Banach space.

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