MATHEMATICAL TRIPOS PART II (2024–2025)
GRAPH THEORY
EXAMPLE SHEET 4 OF 4

1 Show that $R(3,4) \leq 9$. By considering the graph on $\mathbb{Z}_8$ (the integers modulo 8) in which x is joined to y if $x - y = \pm 1$ or ± 2, show that $R(3,4) = 9$.

2 By considering the graph on $\mathbb{Z}_{17}$ in which x is joined to y if $x - y$ is a square modulo 17, show that $R(4,4) = 18$.

3 Show that $R_3(3,3,3) \leq 17$.

4 Let A be a set of $R^{(4)}(n,5)$ points in the plane, with no three points of A collinear. Prove that A contains n points forming a convex n-gon.

5 Let A be an infinite set of points in the plane, with no three points of A collinear. Prove that A contains an infinite set B such that no point of B is a convex combination of other points of B.

6 Show that every graph G has a partition of its vertex-set as $X \cup Y$ such that the number of edges from X to Y is at least $\frac{1}{2}e(G)$. Give three proofs: by induction, by choosing an optimal partition, and by choosing a random partition. Show also that there is a partition of the vertex-set as $X \cup Y$ such that each of $e(X)$ and $e(Y)$ is at most $\frac{1}{3}e(G)$.

7 In a *tournament* on n players, each pair play a game, with one or other player winning (there are no draws). Prove that, for any k, there is a tournament in which, for any k players, there is a player who beats all of them. [Hint: consider a random tournament.] Exhibit such a tournament for $k = 2$.

8 Let X denote the number of copies of K_4 in a random graph G chosen from $G(n,p)$. Find the mean and the variance of X. Deduce that $p = n^{-2/3}$ is a threshold for the existence of a K_4, in the sense that if $pn^{2/3} \to 0$ then almost surely G does not contain a K_4, while if $pn^{2/3} \to \infty$ then almost surely G does contain a K_4.

9 Find the eigenvalues of K_n. Find the eigenvalues of $K_{n,m}$.

10 Prove that the matrix J (all of whose entries are 1) is a polynomial in the adjacency matrix of a graph G if and only if G is regular and connected.

11 Let G be a graph in which every edge is in a unique triangle and every non-edge is a diagonal of a unique 4-cycle. Show that G is k-regular, for some k, and that the number of vertices of G is $1 + k^2/2$. Show also that k must belong to the set $\{2, 4, 14, 22, 112, 994\}$.

12 Let the infinite subsets of $\mathbb{N}$ be 2-coloured. Must there exist an infinite set $M \subseteq \mathbb{N}$ all of whose infinite subsets have the same colour?

Further problems

13 The group of all isomorphisms from a graph G to itself is called the *automorphism group* of G. Show that every finite group is the automorphism group of some graph. Is every group the automorphism group of some (possibly infinite) graph?

SM, Michaelmas Term 2024
Comments on and corrections to this sheet may be emailed to `sm137@cam.ac.uk`