MATHEMATICAL TRIPOS PART II (2024–2025)
GRAPH THEORY
EXAMPLE SHEET 3 OF 4

1 What is the chromatic number of the Petersen graph? What is its edge-chromatic number?

2 Let G be a graph with chromatic number k. Show that $e(G) \geqslant \binom{k}{2}$.

3 Show that, for any graph G, there is an ordering of the vertices of G for which the greedy algorithm uses only $\chi(G)$ colours.

4 For each $k \geq 3$, find a bipartite graph G, with an ordering $v_1, v_2, \ldots, v_n$ of its vertices, for which the greedy algorithm uses k colours. Give an example with $n = 2k - 2$. Is there an example with $n = 2k - 3$?

5 What is $\chi'(K_{n,n})$? What is $\chi'(K_n)$?

6 Let G be a bipartite graph of maximum degree Δ. Must we have $\chi'(G) = \Delta$?

7 Find the chromatic polynomial of the n-cycle.

8 Let G be a graph on n vertices, with $P_G(t) = t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \ldots + a_1 t + a_0$. Show that the a_i alternate in sign (in other words, $a_i \leq 0$ if $n - i$ is odd and $a_i \geq 0$ if $n - i$ is even). Show also that if G has m edges and c triangles then $a_{n-2} = \binom{m}{2} - c$.

9 An *acyclic orientation* of a graph G is an assignment of a direction to each edge of G in such a way that there is no directed cycle. Show that the number of acyclic orientations of G is precisely $|P_G(-1)|$.

10 Let G be a plane graph in which every face is a triangle. Show that the faces of G may be 3-coloured, unless $G = K_4$.

11 Can $K_{4,4}$ be drawn on the torus? What about $K_{5,5}$?

12 A *minor* of a graph G is any graph that may be obtained from a subgraph of G by successively contracting edges – equivalently, a graph H on vertex-set $\{v_1, \ldots, v_r\}$ is a minor of G if we can find disjoint connected subgraphs $S_1, \ldots, S_r$ of G such that whenever $v_i v_j \in E(H)$ there is an edge from S_i to S_j. Show that for any k there is an n such that every graph G with $\chi(G) \geq n$ has a K_k minor. Writing $c(k)$ for the least such n, show that $c(k+1) \leq 2c(k)$. [Hint: choose $x \in G$, and look at the sets $\{y \in G : \; d(x,y) = t\}$.] Show that $c(k) = k$ for $1 \leq k \leq 4$, and explain why $c(5) = 5$ would imply the 4-Colour Theorem.

Further problems

13 Let G be a countable graph in which every finite subgraph can be k-coloured. Show that G can be k-coloured.

14 [Harder] Construct a triangle-free graph of chromatic number 1526.

SM, Michaelmas Term 2024
Comments on and corrections to this sheet may be emailed to sm137@cam.ac.uk