GRAPH THEORY - EXAMPLE SHEET 1

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(1) Show that a graph G which contains an odd circuit, contains an odd cycle.

(2) Show there are infinitely many planar graphs for which $e(G) = 3(|G| - 2)$. Can you give a nice description of all graphs that satisfy this equality?

(3) Show that every graph G, with $|G| \geqslant 2$, has two vertices of the same degree.

(4) Show that in every connected graph $|G| > 2$ there exists a vertex so that $G - v$ is connected.

(5) Show that if G is an acylic and $|G| \geqslant 1$ then $e(G) \leqslant n - 1$.

(6) The *degree sequence* of a graph $G = (\{x_1, \dots, x_n\}, E)$ is the sequence $d(x_1), \dots, d(x_n)$.
For $n \geqslant 2$ let $1 \leqslant d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$ be integers. Show that $(d_i)_{i=1}^n$ is a degree sequence of a tree if and only if $\sum_{i=1}^n d_i = 2n - 2$.

(7) Let $T_1, \dots, T_k$ be subtrees of a tree T. Show that if $V(T_i) \cap V(T_j) \neq \emptyset$ for all $i, j \in [k]$ then $V(T_1) \cap \cdots \cap V(T_k) \neq \emptyset$.

(8) The *average degree* of a graph $G = (V, E)$ is $n^{-1} \sum_{x \in V} d(x)$. Show that if the average degree of G is d then G contains a subgraph with minimum degree $\geqslant d/2$.

(9) Say that a graph $G = (V, E)$ can be *decomposed* into cycles if E can be partitioned $E = E_1 \cup \ldots \cup E_k$, where each E_i is the edge set of a cycle. Show that G can be decomposed into cycles if and only if all degrees of G are even.

(10) The *clique number* of a graph G is the largest t so that G contains a complete graph on t vertices. Show that the possible clique numbers for a regular graph on n vertices are $1, 2, \dots, \lfloor n/2 \rfloor$ and n.

(11) Show that the Petersen graph (look it up) is non-planar.

(12) Let $G = (V, E)$ be a graph. Show that there is a partition $V = A \cup B$ so all the vertices in the graphs $G[A]$ and $G[B]$ are of even degree.

(13) An $n \times n$ Latin square (resp. $r \times n$ Latin rectangle) is an $n \times n$ (resp. $r \times n$) matrix, with each entry from $\{1, \dots, n\}$, such that no two entries in the same row or column are the same. Prove that every $r \times n$ Latin rectangle may be extended to an $n \times n$ Latin square.

(14) (*) Let $G = (X \cup Y, E)$ be a countably infinite bipartite graph with the property that $|N(A)| \geqslant |A|$ for all $A \subseteq X$. Give an example to show that G need not contain a matching from X to Y. On the other hand, show that if all of the degrees of G are finite then G *does* contain a matching from X to Y. Does this remain true if G is uncountable and all degrees of X are finite (while degrees in Y have no restriction)?

(15) (*) Let $M = (X, d_M)$ be a metric space. We say that a function $f : X \to \mathbb{R}^2$ has distortion $\leqslant D$ if there exists an $r > 0$ so that

$$r d_M(x, y) \leqslant \|f(x) - f(y)\|_2 \leqslant D r d_M(x, y).$$

Show that there is a a metric space $M = (\{x_1, \dots, x_n\}, d_M)$ on n points so that every function $f : M \to \mathbb{R}^2$ has distortion $\geqslant c n^{1/2}$, for some constant $c > 0$. Does there exist a metric space $M = (\{x_1, \dots, x_n\}, d_M)$ on n points so that every function $f : M \to \mathbb{R}^2$ has distortion $\geqslant cn$, for some constant $c > 0$?