GRAPH THEORY - EXAMPLE SHEET 4

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(1) For $p \gg n^{-2}$, let $G \sim G(n,p)$. Show that for all $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mathbb{P}\left(|e(G) - p\binom{n}{2}| > \varepsilon p n^2\right) = 0.$$

Use this to finish the sketch seen in class of the fact that $Z(n,t) \geqslant (1/2)n^{2-2/t}$, for sufficiently large n. [Of course, we also saw a proof of the better bound $Z(n,t) > (1/2)n^{2-2/(t+1)}$, but this was using the "alteration method".]

(2) Let $G \sim G(n,p)$. Show that

$$\lim_{n\to\infty} \mathbb{P}(G \supset K_4) = \begin{cases} 1 \text{ if } p \gg n^{-2/3} \\ 0 \text{ if } p \ll n^{-2/3} \end{cases}$$

What is the corresponding statement if we replace K_4 with K_r?

(3) Let H be the graph on 4 vertices x_1, x_2, x_3, x_4 where x_1, x_2, x_3 form a triangle, $x_4x_1 \in E(H)$ and there are no other edges. Let $G \sim G(n,p)$. Show that $\lim_{n\to\infty} \mathbb{P}(G \supset H) = 1$ when $p \gg n^{-1}$.

(4) Give an example of a connected graph H and a $p = p(n)$ so that the expected number of copies of H in $G \sim G(n,p)$ tends to infinity as $n \to \infty$ *but* $\mathbb{P}(G \supset H) \leqslant 1/2$, for all sufficiently large n.

(5) A *dominating set* in a graph G is a set $S \subseteq V(G)$ with the property that every vertex $v \in V(G) \setminus S$ is adjacent to a vertex in S. For $k \geqslant 3$, let G be a k-regular graph. Show that there is a dominating set of size at most $10n(\log k)/k$ in G.

(6) Calculate the eigenvalues of K_n and $K_{m,n}$.

(7) Prove that the matrix J (all of whose entries are 1) is a polynomial in the adjacency matrix of a graph G if and only if G is regular and connected.

(8) Let G be a graph in which every edge is in a unique triangle and every non-edge is a diagonal of a unique 4-cycle. Show that G is k-regular, for some k, and that the number of vertices of G is $k^2/2 + 1$. Show also that $k \in \{2, 4, 14, 22, 112, 994\}$.

(9) Let $G(\mathbb{N}, 1/2)$ be the random graph defined on vertex set $\mathbb{N}$ where every edge is included independently with probability 1/2. If two graphs G_1, G_2 are drawn independently from $G(\mathbb{N}, 1/2)$ show that G_1 is isomorphic to G_2 with probability 1.

(10) Let $S \subset [0,1] \times [0,1]$ be a finite set of points. Define $T(S)$ to be the minimum volume of any triangle formed by some three distinct points $x, y, z \in S$. For all $n \geqslant 3$, show that there is a set of n points $S_n \subset [0,1] \times [0,1]$ with $T(S_n) \geqslant c/n^2$, where $c > 0$ is an absolute constant.

(11) Show that there exists $c > 0$ so that the following holds. Let $p = n^{-2/3}$ and $G \sim G(n,p)$. Then G contains $\geqslant cn$ vertex-disjoint triangles with probability tending to 1 as $n \to \infty$.

(12) (*) Can the edges of K_{10} be decomposed into 3 disjoint copies of the Petersen graph?