GRAPH THEORY - EXAMPLE SHEET 3

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(1) By considering the graph on $\mathbb{Z}_{17}$ in which x is joined to y if $x - y$ is a square modulo 17, show that $R(4) = 18$.

(2) Prove that $R_3(3,3,3) \leqslant 17$.

(3) Let $\{I_i\}_{i \in \mathbb{N}}$ be a countable collection of closed, non-empty, finite intervals in $\mathbb{R}$. Show that either there exists an infinite collection of pairwise disjoint intervals *or* there exists a $x \in \mathbb{R}$ that is covered by infinitely many intervals.

(4) Prove the following statement. For every $t \geqslant 3$, there exists a $n_0(t)$ so that every $S \subseteq \mathbb{R}^2$ with (a) $|S| \geqslant n_0(t)$ (b) no three points on a line contains $x_1, \ldots, x_t \in S$ which are in convex position.

(5) A *tournament* is an oriented complete graph: every edge $\{u, v\} \in E(K_n)$ is given an unique direction, (u, v) or (v, u). Show that a tournament contains a*directed* path of length $n-1$: a sequence of *distinct* vertices $x_1 x_2, \ldots, x_n$ so that $(x_i, x_{i+1}) \in E$ for all $1 \leqslant i \leqslant n-1$.

(6) Show that $R_k(3, \ldots, 3) \leqslant \lceil e \cdot k! \rceil$, where $e = 2.718....$ is Euler's number.

(7) Show that if G is a graph then there exists a bipartite subgraph of G with at least $e(G)/2$ edges. Use this to show that for all $t \in \mathbb{N}$ there exists a $c > 0, n_0(t) \in \mathbb{N}$ so that

$$\mathrm{ex}\,(n, K_{t,t}) \leqslant cn^{2-1/t},$$

for all $n \geqslant n_0(t)$.

(8) Given n distinct points $x_1, \ldots, x_n \in \mathbb{R}^2$ and n distinct lines $\ell_1, \ldots, \ell_n \subseteq \mathbb{R}^2$, show that there are at most $10n^{3/2}$ point-line incidences, that is

$$|\{(i,j) : x_i \in \ell_j\}| < 10n^{3/2}.$$

(9) An *independent set* in a graph is a collection of vertices where no two are adjacent. Show that if $G = (V, E)$ is a graph then there is an independent set $I \subseteq V$ with

$$|I| \geqslant \sum_{x \in V} \frac{1}{d(x)+1}.$$

Now use this result to give another proof of Turán's Theorem. [Hint: for the first part, consider a random ordering of V]

(10) For $\varepsilon > 0$, let G be a graph with $|G| = n$ and $e(G) > (1+\varepsilon)n^2/4$ edges. Show that G contains at least $c\varepsilon n^3$ triangles, for some constant $c > 0$. Use this to give a different proof of the Erdős-Stone theorem in the case $\chi(H) = 3$.

(11) A bow-tie is the graph on 5 vertices consisting of two triangles which share a single vertex. Show that if $|G| > 5$ and $e(G) > \frac{n^2}{4} + 1$ then G contains a bow-tie.

(12) (*) Let G be a 3-regular graph and let $e \in E$. Show that G has an even number of Hamilton cycles through e (quick reminder: 0 is an even number).

(13) (*) Among a group of n dons, any two have exactly one mutual friend. Show that some don is friends with all the others