GRAPH THEORY - EXAMPLE SHEET 2

Michaelmas 2022

Julian Sahasrabudhe

(1) For a graph G, show that $\kappa(G) \leqslant \lambda(G) \leqslant \delta(G)$.

(2) Let G be a graph. Show that $e(G) \geqslant \binom{\chi(G)}{2}$.

(3) Let G be a k-connected graph and let $y, x_1, \ldots, x_k$ be distinct vertices in G. Show that there exists paths $P_1, \ldots, P_k$, where P_i is a $y - x_i$ path and $P_1, \ldots, P_k$ have no vertices in common, apart from the vertex y.

(4) An *independent set* in a graph $G = (V, E)$ is a subset $I \subset V$ so that $x \not\sim y$ for all $x, y \in I$. Let $G = (V, E)$ be a connected graph with $\Delta(G) \leqslant 3$ and $|V| \geqslant 10$. Show that there exists an independent set $I \subseteq V$ so that every odd cycle in G intersects I.

(5) Determine the chromatic polynomial of the n-cycle C_n.

(6) Let G be a graph on n vertices, show that the coefficients of the the chromatic polynomial P_G alternate in sign. That is, if $P_G = \sum_{i=0}^{n} c_i t^i$, Then $c_{n-j} \geqslant 0$ for even j and $c_{n-j} \leqslant 0$ for odd j. Also show that if G has m edges and k triangles then $c_{n-2} = \binom{m}{2} - k$.

(7) Determine $\chi'(K_{n,n})$. Determine $\chi'(K_n)$.

(8) Let G be a graph that has an orientation where the longest directed path has length t (that is, a sequence of oriented edges $(v_1, v_2), \ldots, (v_t, v_{t+1})$). Then $\chi(G) \leqslant t + 1$.

(9) Can $K_{4,4}$ be drawn on the torus? What about $K_{5,5}$?

(10) Let G be a bipartite graph with maximum degree Δ. Must we have $\chi'(G) = \Delta(G)$?

(11) Let $G = (V, E)$ be a graph where V, E are countably infinite. Show that $\chi(G) \leqslant k$ if and only if $\chi(H) \leqslant k$ for every finite subgraph H of G.

(12) For $k \geqslant 2$, let $G = (V, E)$ be a k-connected graph and let $\{x_1, \ldots, x_k\} \subseteq V$. Show that there exists a cycle containing each of the vertices $x_1, \ldots, x_k$.

(13) For each $r \geqslant 2$, construct a graph G that does not contain a K_{r+1} and $\chi(G) > r$.

(14) A graph is *outer-planar* if it can be drawn in the plane so that all of its vertices are on the infinite face. Articulate a conjecture of the form "Let G be a graph with $|G| \geqslant 5$. G is outer-planar if and only if ....". Prove your conjecture.

(15) (*) Show there is a triangle free graph with chromatic number 2022.

(16) (*) Let G be a triangulation (a plane graph where every face is a triangle) and let G° be the *planar dual* of G: the vertices of G° are the faces of G and edges in G° join faces that share a boundary edge (in G). Prove that $\chi(G) \leqslant 4$ if and only if $\chi'(G^\circ) \leqslant 3$.