GRAPH THEORY - EXAMPLE SHEET 3

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(1) Let G be a graph with n vertices. Show that if $e(G) > \binom{n}{2} - (n-2)$ then G contains a Hamilton cycle.

(2) Find the chromatic polynomial of the n cycle.

(3) Is the Petersen graph planar?

(4) Let G be a graph and let $P_G(t) = \sum_i c_i t^i$ be its chromatic polynomial. Show that the c_i alternate in sign (i.e. $c_i \leqslant 0$ if $n-i$ is odd and $c_i \geqslant 0$ if $n-i$ is even). Show also that if G has m edges and r triangles then $c_{n-2} = \binom{m}{2} - r$.

(5) Given n distinct points $x_1, \dots, x_n \in \mathbb{R}^2$ and n distinct lines $\ell_1, \dots, \ell_n \subseteq \mathbb{R}^2$, show that there are at most $10n^{3/2}$ point-line incidences, that is

$$|\{(i,j) : x_i \in \ell_j\}| < 10n^{3/2}.$$

(6) Show every graph with $\delta(G) \geqslant 3$ contains a subdivision of K_4.

(7) An acylic orientation of a graph G is an assignment of an orintation to each edge (i.e. $uv \to (u,v)$ or $uv \to (v,u)$) so that there is no directed cycle. Let G be a graph and $P_G(t)$ be its chromatic polynomal. Show that $|P_G(-1)|$ is the number of acylic orientations of the edges of G.

(8) Can $K_{4,4}$ be drawn on the torus? What about $K_{5,5}$?

(9) A *tournament* is an oriented complete graph: every edge $\{u,v\}$ is given an unique direction, (u,v) or (v,u). Show that a tournament contains a *directed* path of length $n-1$: a sequence of vertices $x_1 x_2, \dots, x_n$ so that $(x_i, x_{i+1}) \in E$ for all $1 \leqslant i \leqslant n-1$.

(10) In a tournament T, a *king* is a vertex v which can reach any other vertex via a directed path of length at most 2. Show that every tournament has a king.

(11) A bow-tie is the graph on 5 vertices consisting of two triangles which share a single vertex. Show that if $|G| > 5$ and $e(G) > \frac{n^2}{4} + 1$ then G contains a bow-tie.

(12) An *independent set* in a graph is a collection of vertices where no two are adjacent. Show that if $G = (V,E)$ is a graph then there is an independent set $I \subseteq V$ with

$$|I| \geqslant \sum_{x \in V} \frac{1}{d(x)+1}.$$

[Hint: consider a random ordering of V]

(13) (*) Let G be a 3-regular graph and let $e \in E$. Show that G has an even number of Hamilton cycles through e (quick reminder: 0 is an even number).

(14) (*) Among a group of n dons, any two have exactly one mutual friend. Show that some don is friends with all the others